Evaluations of the twisted Alexander polynomials of 2-bridge knots at ± 1

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ABSTRACT

Let $H(p)$ be the set of 2-bridge knots $K(r), 0 < r < 1$, such that the group $G(K(r))$ of $K(r)$ is mapped onto a non-trivial free product, $\mathbb{Z}/2 * \mathbb{Z}/p$, p being odd. Then there is an algebraic integer s_0 such that for any $K(r)$ in $H(p)$, $G(K(r))$ has a parabolic representation ρ into $SL(2, \mathbb{Z}[s_0]) \subset SL(2, \mathbb{C})$. Let $\widetilde{\Delta}_{\rho,K(r)}(t)$ be the twisted Alexander polynomial associated to ρ. Then we prove that for any $K(r)$ in $H(p)$, $\widetilde{\Delta}_{\rho,K(r)}(1) = -2s_0^{-1}$ and $\widetilde{\Delta}_{\rho,K(r)}(-1) = -2s_0^{-1}\mu^2$, where $s_0^{-1}, \mu \in \mathbb{Z}[s_0]$. The number μ can be recursively evaluated.

Keywords: Alexander polynomial, 2-bridge knot, knot group, parabolic representaion, twisted Alexander polynomial, continued fraction.

1. Introduction and statement of the main theorem

The twisted Alexander polynomial of a knot K is a significant generalization of the classical Alexander polynomial of K [12] and so far, many attempts have been made to prove that both polynomials share certain important properties [6], [7], [2], [3], [5]. However, such a generalization is by no means straightforward. In fact, there are only few studies on the corresponding question to one of the fundamental properties of the Alexander polynomial : $\Delta_K(1) = 1$ [13]. In this paper, we give some information on the twisted Alexander polynomials of 2-bridge knots evaluated at $t = 1$ and -1. To be more precise, given an odd integer p, let $K(r)$ $r \in \mathbb{Q}, 0 < r < 1$, be a 2-bridge knot such that $G(K(r))$, the group of $K(r)$, is mapped onto a non-trivial free product, $\mathbb{Z}/2 * \mathbb{Z}/p$ and $H(p)$ the set of all 2-bridge knots with this property. Then there is an algebraic integer s_0 such that the group of each knot $K(r)$ in $H(p)$ has a parabolic representation ρ in $SL(2, \mathbb{Z}[s_0]) \subset SL(2, \mathbb{C})$ defined by $\rho : x \mapsto \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

and $y \mapsto \begin{bmatrix} 1 & 0 \\ s_0 & 1 \end{bmatrix}$ where x and y are meridian generators of $G(K(r))$. Let $\widetilde{\Delta}_{\rho,K(r)}(t)$ be the twisted Alexander polynomial of $K(r)$ associated to ρ. Then we prove:

Theorem A. *For any knot $K(r)$ in $H(p)$, we have:*
(1) $\widetilde{\Delta}_{\rho,K(r)}(1) = -2s_0^{-1}$, and
(2) $\widetilde{\Delta}_{\rho,K(r)}(-1) = -2s_0^{-1}\mu^2$,
where both s_0^{-1} and μ are elements of $\mathbb{Z}[s_0]$.

In particular, $K(1/p)$ belongs to $H(p)$ and since for any knot $K(r)$ in $H(p)$, there is an epimorphism from $G(K(r))$ in $H(p)$ to $G(K(\frac{1}{p}))$, it follows that $\widetilde{\Delta}_{\rho,K(1/p)}(t)$ divides $\widetilde{\Delta}_{\rho,K(r)}(t)$ ([9] or see Proposition 3.2(2)), and the quotient $\lambda_{\rho,K(r)}(t) = \widetilde{\Delta}_{\rho,K(r)}(t)/\widetilde{\Delta}_{\rho,K(1/p)}(t)$ is a symmetric polynomial over $\mathbb{Z}[s_0]$ (Proposition 3.2 (3)). Then Theorem A, Proposition 2.4 and (4.3)(2) imply that $\lambda_{\rho,K(r)}(1) = 1$ and $\lambda_{\rho,K(r)}(-1) = \mu^2$, for $\mu \in \mathbb{Z}[s_0]$. If $p = 3$, then $s_0 = -1$, and hence $\lambda_{\rho,K(r)}(t)$ is the Alexander polynomial $\Delta_K(t)$ of some knot K. However, the second condition of Theorem A gives a strong restriction for $\Delta_K(t)$. Therefore, for example, the quadratic Alexander polynomial cannot be realized as the polynomial $\lambda_{\rho,K(r)}(t)$ for any knot $K(r)$, since the degree of $\lambda_{\rho,K(r)}(t)$ must be a multiple of 4 (Proposition 3.4). On the other hand, for some particular r, $\lambda_{\rho,K(r)}(t)$ can be realized as the Alexander polynomial. In fact, we can prove:

Proposition 3.5. *For any odd integers p and q, $\lambda_{\rho,K(1/pq)}(t) = \Delta_{K(1/q)}(t^{2p})$.*

The number $\mu \in \mathbb{Z}[s_p]$ is a knot invariant, and μ can easily be evaluated by using a recursion formula. (See Proposition 9.1.)

After the first draft of the present paper was completed, we learned that D. Silver and S. Williams have been studying a similar problem with a different motivation and they propose a quite interesting conjecture that is closely related to Theorem A. As an application of Theorem A, we prove their conjecture partially for 2-bridge knots in $H(p)$ in Section 10.

This paper is organized as follows. In Section 2, first we give a quick review of the definition of the twisted Alexander polynomial and state their basic properties. Then we define a parabolic representation of a 2-bridge knot $K(r)$ and for a few values of r, we calculate the twisted Alexander polynomial of $K(r)$ associated to this representation. In Section 3, we introduce a polynomial $\lambda_{\rho,K(r)}(t)$ for $K(r)$ when $G(K(r))$ is mapped onto the free product $\mathbb{Z}/2 * \mathbb{Z}/p, p$ being odd, and determine $\lambda_{\rho,K(r)}(t)$ for some values r. In Sections 4, we introduce a $\mathbb{Z}[s_0]$-algebra $\widetilde{A}(s_0)$ that is our fundamental tool to prove Theorem A, and verify two technical lemmas about $\widetilde{A}(s_0)$. In Section 5, as the first step toward the proof of Theorem A, we show that Theorem A is reduced to two formulas in the algebra $\widetilde{A}(s_0)$. The purpose of the next section, Section 6, is to show that we only need to prove Theorem A for much

restricted rationals r. (See Propositions 6.3 and 8.1.) In Section 7, we prove the first part of Theorem A, and the second part of Theorem A is proved in Section 8. In Section 9, we provide an algorithm to evaluate the number μ appeared in Theorem A. In the last section, Section 10, we state Silver-Williams Conjecture and prove their conjecture for torus knots $K(1/p), p$ odd, and for 2-bridge knots in $H(p)$. In Appendix, we give an outline of the proofs of Proposition 2.4 and (10.4)(2), and also give a proof of Proposition 3.5.

2. Definition and Examples

In this section, first we quickly review the definition of the twisted Alexander polynomials and their properties that we will use throughout this paper. For the details, we refer to [19]. Later in this section, we define a parabolic representation of the group of a 2-bridge knot $K(r)$. (See [15].)

Let $\rho : G = G(K) \to GL(n, \mathbb{C})$ be a linear representation of the group of a knot K. Let $G = \langle x_1, x_2, \cdots, x_m | r_1, r_2, \cdots, r_{m-1}\rangle$ be a Wirtinger presentation of $G(K)$. Denote by $M_{p,q}(R)$ the ring of $p \times q$ matrices over a ring R. Let $A = \left[\frac{\partial r_i}{\partial x_j}\right] \in M_{m-1,m}(\mathbb{Z}[x_1^{\pm 1}, \cdots, x_m^{\pm 1}])$ be the Alexander matrix, where $\frac{\partial}{\partial x_j}$ denotes Fox free derivatives and $\mathbb{Z}[x_1^{\pm 1}, \cdots, x_m^{\pm 1}]$ is a non-commutative ring of Laurent polynomials. The square matrix $\widehat{A} = \left[\frac{\partial r_i}{\partial x_j}\right]_{1 \le i,j \le m-1}$ is obtained by deleting the last column of A. We define a homomorphism Φ from the group ring $\mathbb{Z}G$ into $M_{n,n}(\mathbb{C}[t^{\pm 1}])$ by $\Phi(x_i) = \rho(x_i)t$. Then $(\frac{\partial r_i}{\partial x_j})^{\Phi} \in M_{n,n}(\mathbb{C}[t^{\pm 1}])$, and hence $(\widehat{A})^{\Phi} = \left[(\frac{\partial r_i}{\partial x_j})^{\Phi}\right] \in M_{(m-1)n,(m-1)n}(\mathbb{C}[t^{\pm 1}])$.

Definition 2.1. [19] The twisted Alexander polynomial of K associated to ρ is defined as follows:

$$\widetilde{\Delta}_{\rho,K}(t) = \frac{\det \widehat{A}^{\Phi}}{\det(x_m^{\Phi} - 1)} \in \mathbb{C}[t^{\pm 1}].$$

If ρ is unimodular, this is an invariant of K up to $\pm t^{nk}$.

We should note that for any linear representation ρ, the ambiguity of this invariant is completely eliminated by Kitayama. For the precise formulation, see [10].

Remark 2.2. (1) If $\rho : x_i \mapsto I \in GL(n, \mathbb{C})$ is a trivial representation, then $\widetilde{\Delta}_{\rho,K}(t) = \left[\frac{\Delta_K(t)}{t-1}\right]^n$, where $\Delta_K(t)$ is the Alexander polynomial of a knot K. (2) In general, $\widetilde{\Delta}_{\rho,K}(t)$ is a rational function, but it is shown [19] that if the commutator subgroup G' contains an element w such that 1 is not an eigenvalue of $\rho(w)$, then $\widetilde{\Delta}_{\rho,K}(t)$ is a Laurent polynomial over $\mathbb{C}$, namely, $\widetilde{\Delta}_{\rho,K}(t) \in \mathbb{C}[t^{\pm 1}]$. (3) For any presentation $\rho : G(K) \to SL(n, \mathbb{C})$, $\widetilde{\Delta}_{\rho,K}(t)$ is symmetric [8].

Now we study parabolic representations of the 2-bridge knot groups (c.f. [15]). Let r be a rational number, $0 < r = \frac{\beta}{\alpha} < 1$, where both α and β are odd and $\gcd(\alpha, \beta) = 1$, and $K(r)$ is the 2-bridge knot of type (α, β).

Let $F(x, y)$ be the free group freely generated by x and y. For $k = 1, 2, \cdots, \alpha-1$, let $\eta_k = [\frac{k\beta}{\alpha}]$, where $[\cdot]$ denotes Gaussian symbol and let $\varepsilon_k = (-1)^{\eta_k}$.

Using the word W given by

$$W = x^{\epsilon_1} y^{\epsilon_2} x^{\epsilon_3} y^{\epsilon_4} \cdots x^{\epsilon_{\alpha-2}} y^{\epsilon_{\alpha-1}}, \tag{2.1}$$

we obtain a Wirtinger presentation of $G(K(r))$:

$$G(K(r)) = \langle x, y | WxW^{-1}y^{-1} = 1 \rangle. \tag{2.2}$$

For each $r, 0 < r < 1$, there is a non-commutative representation $\rho : G(K(r)) \to SL(2, \mathbb{C})$ such that

$$\rho(x) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \text{ and } \rho(y) = \begin{bmatrix} 1 & 0 \\ s_r & 1 \end{bmatrix}, s_r \neq 0. \tag{2.3}$$

Here a complex number s_r is determined as follows [15]. Let $G = G(K(r)) = \langle x, y | WxW^{-1}y^{-1} = 1 \rangle$ be a Wirtinger presentation of G given by (2.2).
Set $\rho(x) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ and $\rho(y) = \begin{bmatrix} 1 & 0 \\ z & 1 \end{bmatrix}$, where z is a variable.
Compute $\rho(W) = \begin{bmatrix} a(z) & b(z) \\ c(z) & d(z) \end{bmatrix}$, where a, b, c and d are polynomials on z. Then equality $Wx = yW$ yields $\begin{bmatrix} a & a+b \\ c & c+d \end{bmatrix} = \begin{bmatrix} a & b \\ za+c & zb+d \end{bmatrix}$.
The number s_r we sought is a root of $a(z) = 0$ [15, Theorem 2]. For convenience, we call ρ a *canonical representation* of $G(K(r))$, and $a(z)$ the *representation polynomial* of ρ. Since $G' \ni xy^{-1}$ and $\rho(xy^{-1}) = \begin{bmatrix} 1-s_r & 1 \\ -s_r & 1 \end{bmatrix}$, 1 is not an eigenvalue of $\rho(xy^{-1})$, and by Remark 2.2 (2) and (3), we see that $\widetilde{\Delta}_{\rho,K}(t)$ is a symmetric Laurent polynomial over $\mathbb{Z}[s_r]$. It is known [15] that the representation polynomial $a(z)$ is a separable polynomial of degree $\frac{\alpha-1}{2}$.

If $r = 1/p, p = 2n+1$. Then W $= (xy)^n$, and it is easy to show that the representation polynomial $a_n(z)$ is a monic polynomial of degree n and further, the constant term is also 1. We study $a_n(z)$ in Section 10.

Example 2.3. (1) Let $r = 1/3$. Then $W = xy$ and hence $s_r = -1$. Therefore, $\rho : x \mapsto \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ and $y \mapsto \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$ gives a parabolic representation $\rho : G(K(1/3)) \to SL(2, \mathbb{Z})$. A simple computation shows that $\widetilde{\Delta}_{\rho,K(1/3)}(t) = 1 + t^2$.

(2) Let $r = 3/5$. Then $W = xy^{-1}x^{-1}y$, and hence $s_r = -w$, where w is a primitive cubic root of 1. Thus we have a parabolic representation $\rho : G \to SL(2, \mathbb{Z}[w]) \subset SL(2, \mathbb{C})$ and the twisted Alexander polynomial of $K(3/5)$ associated to ρ is $\widetilde{\Delta}_{\rho,K(3/5)}(t) = 1 - 4t + t^2$.

(3) Let $r = 3/7$, then $W = xyx^{-1}y^{-1}xy$ and s_r is a root of $1 + 2z + z^2 + z^3 = 0$. The twisted Alexander polynomial associated to this representation is $\widetilde{\Delta}_{\rho,K(3/7)}(t) = -(4 + s_r^2) + 4t - (4 + s_r^2)t^2$.

Proposition 2.4. *The twisted Alexander polynomial of $K(1/p)$, $p = 2n+1$, associated to a canonical representation ρ is given by*

$$\widetilde{\Delta}_{\rho,K(1/p)}(t) = b_1 + b_2t^2 + b_3t^4 + \cdots + b_nt^{2n-2} + b_nt^{2n} + b_{n-1}t^{2n+2} + \cdots + b_1t^{4n-2},$$

where b_k is the $(1,2)$-entry of $\rho(xy)^k$, and $b_k = \displaystyle\sum_{j=0}^{k-1} \binom{ktj}{2j+1} s_{1/p}^j$ (see [18]).

For a proof, see Appendix (I).

3. Twisted Alexander polynomials of 2-bridge knots

Suppose that there is an epimorphism from $G(K(r))$ to non-trivial free product, $\mathbb{Z}/2 * \mathbb{Z}/p$ for some odd p. Let $H(p)$ be the set of these knots $K(r)$. The following proposition is proved in [4].

Proposition 3.1. *Let $K(r)$ be an element of $H(p)$. We may assume without loss of generality that $0 < r = \dfrac{\beta}{\alpha} < 1$, where $0 < \beta < \alpha$, $\alpha \equiv \beta \equiv\ 1 \pmod 2$ and $\gcd(\alpha, \beta) = 1$. Then the continued fraction of r is of the form:*
$r = [pk_1, 2m_1, pk_2, 2m_2, \ldots, 2m_q, pk_{q+1}]$, where m_i and k_j are non-zero integers.

Here, the continued fraction of r is defined as follows:

$$r = \frac{\beta}{\alpha} = \cfrac{1}{pk_1 - \cfrac{1}{2m_1 - \cfrac{1}{pk_2 - \ddots - \cfrac{1}{2m_q - \cfrac{1}{pk_{q+1}}}}}}$$

A different characterization of continued fractions of r for $K(r)$ in $H(p)$ is given in Appendix (IV).

According to [14], there is an epimorphism φ from $G(K(r)), K(r) \in H(p)$, onto $G(K(1/p))$ sending meridians of $K(r)$ to those of $K(1/p)$. Therefore, the canonical parabolic representation $\rho : G(K(1/p)) \to SL(2, \mathbb{Z}[s_{1/p}]) \subset SL(2, \mathbb{C})$ defined by

$$\rho(x) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \text{ and } \rho(y) = \begin{bmatrix} 1 & 0 \\ s_{1/p} & 1 \end{bmatrix} \tag{3.1}$$

can be extended to a parabolic representation

$$\rho\varphi : G(K(r)) \to G(K(1/p)) \to SL(2, \mathbb{Z}[s_{1/p}]) \subset SL(2, \mathbb{C}) \tag{3.2}$$

and we can define the twisted Alexander polynomials of $K(r)$ and $K(1/p)$ associated to $\rho\varphi$ and ρ, respectively.

First we prove the following;

Proposition 3.2. *[9] Let $\rho\varphi : G(K(r)) \to SL(2, \mathbb{Z}[s_{1/p}]) \subset SL(2, \mathbb{C})$ be the parabolic representation defined by (3.2). Then,*
(1) Both $\widetilde{\Delta}_{\rho,K(1/p)}(t)$ and $\widetilde{\Delta}_{\rho\varphi,K(r)}(t)$ are polynomials over $\mathbb{Z}[s_{1/p}]$, and
(2) $\widetilde{\Delta}_{\rho,K(1/p)}(t)$ divides $\widetilde{\Delta}_{\rho\varphi,K(r)}(t)$.
Write $\widetilde{\Delta}_{\rho\varphi,K(r)}(t) = \lambda_{\rho,K(r)}(t)\widetilde{\Delta}_{\rho,K(1/p)}(t)$. Then,
(3) $\lambda_{\rho,K(r)}(t)$ is a symmetric polynomial over $\mathbb{Z}[s_{1/p}]$, and $\lambda_{\rho,K(r)}(t)$ is unique up to t^{2k}.

Proof. First, (1) follows from Remark 2.2. To prove (2), consider Wirtinger presentations $G(K(1/p)) = \langle x, y | R_0\rangle$ and $G(K(r)) = \langle x, y | R\rangle$. Since an epimorphism φ sends x to x and y to y, it follows that $R = 1$ in $G(K(1/p))$. Therefore, R is written freely as a product of conjugates of R_0 and

$$R \equiv \prod_{k=1}^{m} u_j R_0^{\epsilon_j} u_j^{-1}, \tag{3.3}$$

where $u_j \in F(x, y)$ and $\epsilon_j = \pm 1$, and $A \equiv B$ means that AB^{-1} is equal to the identity of the free group $F(x, y)$. Therefore, $\Phi(\frac{\partial R}{\partial x}) = \sum_{j=1}^{m} \epsilon_j u_j^{\Phi} (\frac{\partial R_0}{\partial x})^{\Phi}$, where $\Phi : \mathbb{Z}F(x, y) \to M_{2,2}(\mathbb{Z}[s_{1/p}][t^{\pm 1}])$.
Now

$$\begin{aligned}
\widetilde{\Delta}_{\rho,K(r)}(t) &= \det(\frac{\partial R}{\partial x}^{\Phi})/\det(y^{\Phi} - I) \\
&= \det\Big[\sum_{j=1}^{m} \epsilon_j u_j^{\Phi}\Big] \det(\frac{\partial R_0}{\partial x})^{\Phi}/\det(y^{\Phi} - I) \\
&= \det\Big[\sum_{j=1}^{m} \epsilon_j u_j^{\Phi}\Big]\Big[\det(\frac{\partial R_0}{\partial x})^{\Phi}/\det(y^{\Phi} - I)\Big] \\
&= \det\Big[\sum_{j=1}^{m} \epsilon_j u_j^{\Phi}\Big]\widetilde{\Delta}_{\rho,K(1/p)}(t).
\end{aligned}$$

This proves (2), and further, we see that

$$\lambda_{\rho,K(r)}(t) = \det\left[\sum_{j=1}^{m} \epsilon_j u_j^{\Phi}\right]. \tag{3.4}$$

By Remark 2.2, $\lambda_{\rho,K(r)}(t)$ is a symmetric polynomial over $\mathbb{Z}[s_{1/p}]$. This proves (3). $\square$

Example 3.3. (1) From Proposition 3.1, we see that there are epimorphisms from $G(K(19/45))$ and $G(K(37/213))$ onto $G(K(1/3))$. Straightforward calculations show that $\lambda_{\rho,K(19/45)}(t) = 25 - 72t + 95t^2 - 72t^3 + 25t^4$ and $\lambda_{\rho,K(37/213)}(t) = 4 - 16t + 28t^2 - 32t^3 + 28t^4 - 16t^5 + 8t^6 - 8t^7 + 4t^8 - 8t^{10} + 16t^{11} - 15t^{12} + 16t^{13} - 8t^{14} + 4t^{16} - 8t^{17} + 8t^{18} - 16t^{19} + 28t^{20} - 32t^{21} + 28t^{22} - 16t^{23} + 4t^{24}$.

We should note that for these examples, Theorem A holds. In fact, $\lambda_{\rho,K(19/45)}(1) = 1$ and $\lambda_{\rho,K(19/45)}(-1) = 289 = 17^2$, and $\lambda_{\rho,K(37/213)}(1) = 1$ and $\lambda_{\rho,K(37/213)}(-1) = 225 = 15^2$.

If $p = 3$, then $s_{1/3} = -1$ and for $K(r) \in H(3)$, $\lambda_{\rho,K(r)}(t)$ is a symmetric integer polynomial (of even degree) and hence, Theorem A (1) and Proposition 3.2 (3) imply that $\lambda_{\rho,K(r)}(t)$ is the Alexander polynomial of some knot. Further, Theorem A (2) gives another condition that must be satisfied by this Alexander polynomial. Then, it is easy to show the following;

Proposition 3.4. *The degree of $\lambda_{\rho,K(r)}(t)$ is a multiple of 4.*

Proof. Write $\lambda_{\rho,K(r)}(t) = \sum_{j=0}^{2m} a_j t^j$. Suppose m is odd, say $m = 2h + 1$. Since $\lambda_{\rho,K(r)}(1) = 1$ and $\lambda_{\rho,K(r)}(-1) = \mu^2$, it follows that $\sum_{j=0}^{2h} 2a_j + a_{2h+1} = 1$ and $\sum_{j=0}^{2h} (-1)^j 2a_j - a_{2h+1} = \mu^2$, and hence, $\sum_{j=0}^{h} 4a_{2j} = 1 + \mu^2$ that is impossible. $\square$

On the other hand, for some special cases, it is possible to identify $\lambda_{\rho,K(r)}(t)$ as the Alexander polynomial of a certain knot. We can prove the following;

Proposition 3.5. *Suppose p and q are odd integer ≥ 3. Then $K(1/pq) \in H(p)$ and $\lambda_{\rho,K(1/pq)}(t) = \Delta_{K(1/q)}(t^{2p})$, where $\Delta_{K(1/q)}(t)$ is the Alexander polynomial of $K(1/q)$. Therefore, $\lambda_{\rho,K(1/pq)}(t)$ is the Alexander polynomial of the $2p$-cable of the torus knot $K(1/q)$.*

A proof will be given in Appendix (II).

Finally, we note that Theorem A is not true for non-rational knots.

Example 3.6. Consider a non-rational knot $K = 8_5$ in the Reidemeister-Rolfsen table. Then $G(K)$ has a Wirtinger presentation, $G(K) = \langle x, y, z | R_1, R_2 \rangle$, where
$R_1 = yxy^{-1}x^{-1}y^{-1}xyxyx^{-1}y^{-1}z^{-1}yxy^{-1}x^{-1}y^{-1}z$ and
$R_2 = yxy^{-1}zyx^{-1}y^{-1}z^{-1}x^{-1}y^{-1}xz$.
It is easy to check that $\rho : x, z \mapsto \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ and $y \mapsto \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$ gives a parabolic representation of $G(K)$ on $SL(2,\mathbb{Z})$, and $\widetilde{\Delta}_{\rho,K}(t) = -(1-t)^2(1+t^2)(1-2t-2t^3-2t^5+t^6)$. And hence, $\lambda_{\rho,K}(t) = -(1-t)^2(1-2t-2t^3-2t^5+t^6)$, and $\lambda_{\rho,K}(1) = 0$ and $\lambda_{\rho,K}(-1) = 2^5$.

We know, $\lambda_{\rho,K}(t)$ is the reduced Alexander polynomial of a 3-component link.

4. $\mathbb{Z}[s_0]$-Algebra

From now on (except for Appendix), we consider exclusively the set $H(p), p = 2n+1$. For simplicity, we use s_0 for $s_{1/p}$.

Let $X = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, and $Y = \begin{bmatrix} 1 & 0 \\ s_0 & 1 \end{bmatrix}$ be elements in $SL(2, \mathbb{Z}[s_0])$.

We define $A(x, y : \mathbb{Z}[s_0])$ as the free algebra over $\mathbb{Z}[s_0]$ constructed from the free group $F(x, y)$. Let $f : A(x, y : \mathbb{Z}[s_0]) \to M_{2,2}(\mathbb{Z}[s_0])$ be an (algebra) homomorphism defined by $f(x) = X$ and $f(y) = Y$. Let $S(x, y) = f^{-1}(0)$ be the kernel of f. Then $\widetilde{A}(s_0) = A(x, y : \mathbb{Z}[s_0])/S(x, y)$ is a non-commutative $\mathbb{Z}[s_0]$-algebra.

Example 4.1. The following elements are typical elements of $S(x, y)$: $(x - 1)^2$, since $(X - I)^2 = 0$, and $(y - 1)^2, (xy)^n x(xy)^{-n} y^{-1} - 1$ and $(xy)^n x - y(xy)^n$.

The purpose of this section is to prove Lemmas 4.5 and 4.6. However, first we need a few technical lemmas.

For any integer $k \geq 0$, we write

$$(XY)^k = \begin{bmatrix} a_k & b_k \\ c_k & d_k \end{bmatrix} \in SL(2, \mathbb{Z}[s_0]), \tag{4.1}$$

where a_k, b_k, c_k and d_k are integer polynomials in s_0. From the definition of s_0, we should note that $a_n = 0$.

Proposition 4.2. *We have the following recursive formulas.*
(I) $a_0 = d_0 = 1$ *and* $b_0 = c_0 = 0$.
(II) $a_1 = 1 + s_0, b_1 = 1, c_1 = s_0$ *and* $d_1 = 1$.
(III) (i) For $k \geq 2$,
(1) $a_k = (2 + s_0)a_{k-1} - a_{k-2}$,
(2) $s_0 b_k = (1 + s_0)a_{k-1} - a_{k-2}$.
(ii) For $k \geq 1$,
(3) $s_0 b_k = a_k - a_{k-1}$,
(4) $s_0 b_k = c_k$,
(5) $a_k = s_0 b_k + d_k$,
(6) $d_k = a_{k-1}$,
(7) $b_k = b_{k-1} + a_{k-1}$,
(8) $c_k + d_k = a_k$,
(9) $a_0 + a_1 + \cdots + a_{k-1} = b_k$.

Proof. (I) and (II) are immediate. To show (III), we use induction on k. For $k = 1$, (3)-(9) are obvious. For $k = 2$, (1)-(9) are also immediate from the definition, since $(XY)^2 = \begin{bmatrix} 1 + 3s_0 + s_0^2 & 2 + s_0 \\ 2s_0 + s_0^2 & 1 + s_0 \end{bmatrix}$. Now, for any $k \geq 2$, (1) and (2) $\to$ (3), and (3) and

(6) $\to$ (5). Further, since (4) and (5) $\to$ (8), and (2) and (3) $\to$ (7) $\to$ (9), it only suffices to prove (1), (2), (4) and (6). Inductively we assume that these formulas hold for k.

Then a computation $(XY)^{k+1} = (XY)^k(XY)$ shows

$$\begin{aligned}
&(i) \quad a_{k+1} = (1+s_0)a_k + s_0 b_k,\\
&(ii) \ \ b_{k+1} = a_k + b_k,\\
&(iii) \ c_{k+1} = (1+s_0)c_k + s_0 d_k,\\
&(iv) \ d_{k+1} = c_k + d_k.
\end{aligned} \tag{4.2}$$

And we see
(1) $a_{k+1} = (1+s_0)a_k + s_0 b_k = (1+s_0)a_k + a_k - a_{k-1} = (2+s_0)a_k - a_{k-1}$,
(2) $s_0 b_{k+1} = s_0 a_k + s_0 b_k = s_0 a_k + (a_k - a_{k-1}) = (s_0+1)a_k - a_{k-1}$,
(6) $d_{k+1} = c_k + d_k = s_0 b_k + a_{k-1} = a_k$,
(4) $c_{k+1} = (1+s_0)c_k + s_0 d_k = c_k + s_0(c_k + d_k) = s_0 b_k + s_0 a_k = s_0 b_{k+1}$.
This proves Proposition 4.2. □

Proposition 4.3. *(1)* $(XY)^n X = Y(XY)^n = \begin{bmatrix} 0 & b_n \\ c_n & 0 \end{bmatrix}$ *and (2)* $(XY)^p = -I$.

Proof. A direct computation shows (1), since $c_n + d_n = a_n = 0$. Also, (2) follows, since $(XY)^p = (XY)^n XY(XY)^n = -I$. □

Note that $\det\Big[(XY)^n X\Big] = -b_n c_n = 1$.

Proposition 4.4. *We have the following equalities:*

$$\begin{aligned}
&(1)\ a_0 + a_1 + \cdots + a_{n-1} = b_n.\\
&(2)\ s_0(b_1 + b_2 + \cdots + b_n) = -1.\\
&(3)\ b_1 + b_2 + \cdots + b_n = b_n^2.\\
&(4)\ d_0 + d_1 + \cdots + d_n = 1 + a_0 + a_1 + \cdots + a_{n-1}.\\
&(5)\ c_1 + c_2 + \cdots + c_n = -1.
\end{aligned} \tag{4.3}$$

Proof. First, (1) follows from Proposition 4.2(9). To show (2), use Proposition 4.2(III)(3). In fact, since $a_n(s_0) = 0$, $\sum_{k=1}^n s_0 b_k = \sum_{k=1}^n (a_k - a_{k-1}) = a_n - a_0 = -1$. (3) follows, since $b_n^2 = b_n c_n s_0^{-1} = -s_0^{-1}$. (4) follows from Proposition 4.2 (III)(6). Finally, since $c_k = s_0 b_k$, it follows that $\sum_{k=1}^n c_k = s_0 \sum_{k=1}^n b_k = -1$, and (5) is proved. □

We proceed to prove two key lemmas below. For simplicity, we use the following

notations:

$$
\begin{aligned}
&(1)\ \boldsymbol{a}=\sum_{k=0}^{n} a_k, \boldsymbol{b}=\sum_{k=0}^{n} b_k, \boldsymbol{c}=\sum_{k=0}^{n} c_k \text{ and } \boldsymbol{d}=\sum_{k=0}^{n} d_k,\\
&(2)\ P_k = 1+(xy)+(xy)^2+\cdots+(xy)^k,\\
&(3)\ Q_k = yP_ky^{-1} = 1+(yx)+(yx)^2+\cdots+(yx)^k.
\end{aligned}
\tag{4.4}
$$

Lemma 4.5. *The following equalities hold in* $\widetilde{A}(s_0)$.

$$
\begin{aligned}
&(1)\ (1-y)Q_ny(1-x) = -(yx)^{n+1}(1-x).\\
&(2)\ (1-y)Q_{2n}y(1-x) = 0.\\
&(3)\ (1-y)Q_{3n+1}y(1-x) = -(yx)^{3n+2}(1-x).
\end{aligned}
\tag{4.5}
$$

Proof. Since $(YX)^k = Y(XY)^kY^{-1}$, it follows that $(YX)^k = \begin{bmatrix} d_k & b_k \\ c_k & a_k \end{bmatrix}$.

Proof of (1). By taking the image of both sides under f, we have
LHS $= \begin{bmatrix} 0 & 0 \\ -s_0 & 0 \end{bmatrix}\begin{bmatrix} \boldsymbol{d} & \boldsymbol{b} \\ \boldsymbol{c} & \boldsymbol{a} \end{bmatrix}\begin{bmatrix} 1 & 0 \\ s_0 & 1 \end{bmatrix}\begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & s_0\boldsymbol{d}+s_0^2\boldsymbol{b} \end{bmatrix}$.
On the other hand, since $x(1-x)=1-x$ and $y(xy)^n=(xy)^nx$, we see $-(yx)^{n+1}(1-x) = -y(xy)^nx(1-x) = -y(xy)^n(1-x)$.
Also $f(y(xy)^n) = \begin{bmatrix} 0 & b_n \\ c_n & 0 \end{bmatrix}$, and hence RHS $= -\begin{bmatrix} 0 & b_n \\ c_n & 0 \end{bmatrix}\begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & c_n \end{bmatrix}$.
Use (4.3) (4) to show $s_0\boldsymbol{d}+s_0^2\boldsymbol{b} = s_0(1+\boldsymbol{a})-s_0 = s_0\boldsymbol{a} = s_0b_n = c_n$. This proves (1).

Proof of (2). Since

$$
\begin{aligned}
(1-y)Q_{2n}y(1-x) &= (1-y)Q_ny(1-x) + (1-y)(Q_{2n}-Q_n)y(1-x)\\
&= -(yx)^{n+1}(1-x) + (1-y)(Q_n-1)(yx)^ny(1-x),
\end{aligned}
$$

it suffices to show that $(1-y)(Q_n-1)(yx)^ny(1-x) = (yx)^{n+1}(1-x)$. Take the image of both sides under f. Then,
LHS $= \begin{bmatrix} 0 & 0 \\ -s_0 & 0 \end{bmatrix}\begin{bmatrix} \boldsymbol{d}-1 & \boldsymbol{b} \\ \boldsymbol{c} & \boldsymbol{a}-1 \end{bmatrix}\begin{bmatrix} 0 & b_n \\ c_n & 0 \end{bmatrix}\begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & -c_n \end{bmatrix}$.
Meanwhile, RHS $= \begin{bmatrix} 0 & 0 \\ 0 & -c_n \end{bmatrix}$, as is shown in the proof of (1). This proves (2).

Proof of (3). Since $(yx)^{2n+1} = -1$, we see that

$$
\begin{aligned}
(1-y)Q_{3n+1}y(1-x) &= (1-y)Q_{2n}y(1-x) + (1-y)(Q_{3n+1}-Q_{2n})y(1-x)\\
&= -(1-y)Q_ny(1-x)\\
&= (yx)^{n+1}(1-x)\\
&= -(yx)^{3n+2}(1-x).
\end{aligned}
$$

$\square$

Lemma 4.6. *The following equalities hold in* $\widetilde{A}(s_0)$.

(1) $(1+y)Q_n y(1+x) = (yx)^{n+1}(1+x) + 4b_n(y+(yx)^{n+1})$.

(2) $(1+y)Q_n y(1+x)(1+(xy)^n x) = (yx)^{n+1}(1+x)(1+(xy)^n x) + 8b_n(yx)^{n+1}$.

(3) $(1+y)Q_{2n} y(1+x) = 8b_n(yx)^{n+1}$.

(4) $(1+y)Q_{2n} y(1+x)(1+(yx)^n y) = -8b_n(y-(yx)^{n+1})$.

(5) $(1+y)Q_{3n+1} y(1+x) + (yx)^{n+1}(1+x) = -4b_n(y-(yx)^{n+1})$. (4.6)

Proof. First we note that (2) follows from (1), and (4) follows from (3) by multiplying both sides through $(1+(xy)^n x)$, since $(y+(yx)^{n+1})(1+(xy)^n x) = y(1+(xy)^n x)^2 = y(1+2(xy)^n x+(xy)^{2n+1}) = 2(yx)^{n+1}$ and $(yx)^{n+1}(1+(yx)^n y) = (yx)^{n+1}+(yx)^{2n+1}y = (yx)^{n+1}-y$. Also (5) follows from (1) and (3), since $Q_{3n+1} = Q_{2n} - Q_n$. Therefore, we only need to show (1) and (3).

Proof of (1). Since LHS $= (1+y)Q_n y(1+x) = (1+y)yP_n(1+x) = y(1+y)P_n(1+x)$, and RHS $= (yx)^{n+1}(1+x) + 4b_n y(1+(xy)^n x) = y(xy)^n x(1+x) + 4b_n y(1+(xy)^n x)$, it suffices to show

(1)' $(1+y)P_n(1+x) = (xy)^n x(1+x) + 4b_n(1+(xy)^n x)$.

By taking the image of both sides of (1)' under f, we obtain

$$\text{LHS} = \begin{bmatrix} 2 & 0 \\ s_0 & 2 \end{bmatrix}\begin{bmatrix} \boldsymbol{a} & \boldsymbol{b} \\ \boldsymbol{c} & \boldsymbol{d} \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 4\boldsymbol{a} & 2\boldsymbol{a}+4\boldsymbol{b} \\ 2s_0\boldsymbol{a}+4\boldsymbol{c} & s_0\boldsymbol{a}+2\boldsymbol{c}+2s_0\boldsymbol{b}+4\boldsymbol{d} \end{bmatrix}, \text{ and}$$

$$\text{RHS} = \begin{bmatrix} 0 & b_n \\ c_n & 0 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix} + 4b_n\begin{bmatrix} 1 & b_n \\ c_n & 1 \end{bmatrix} = \begin{bmatrix} 4b_n & 2b_n+4b_n^2 \\ 2c_n+4b_nc_n & c_n+4b_n \end{bmatrix}.$$

Therefore we need to show

$$\begin{aligned} &(i)\ 4\boldsymbol{a} = 4b_n, \\ &(ii)\ 2\boldsymbol{a}+4\boldsymbol{b} = 2b_n+4b_n^2, \\ &(iii)\ 2s_0\boldsymbol{a}+4\boldsymbol{c} = 2c_n+4b_nc_n, \text{ and} \\ &(iv)\ s_0\boldsymbol{a}+2\boldsymbol{c}+2\boldsymbol{b}s_0+4\boldsymbol{d} = c_n+4b_n. \end{aligned} \tag{4.7}$$

First, (i) follows from (4.3)(1), and (ii) follows from (4.3)(1) and (3). Further, (iii) follows, since $s_0\boldsymbol{a} = s_0b_n = c_n$ and $\boldsymbol{c} = s_0\boldsymbol{b} = -1 = b_nc_n$. Finally, (iv) follows, since $s_0\boldsymbol{a}+2\boldsymbol{c} = c_n-2$, and $2\boldsymbol{b}s_0+4\boldsymbol{d} = -2+4(1+\boldsymbol{a}) = 2+4b_n$ by (4.3) (1) and (5). A proof of (1) is now complete.

Proof of (3). First, we note

LHS $= (1+y)Q_{2n}y(1+x) = (1+y)Q_ny(1+x) + (1+y)(Q_{2n}-Q_n)y(1+x)$.

Since by (4.6)(1), $(1+y)Q_ny(1+x) = (yx)^{n+1}(1+x) + 4b_n(y+(yx)^{n+1})$ and $(1+y)(Q_{2n}-Q_n)y(1+x) = (1+y)(Q_n-1)(yx)^n y(1+x)$, we must show $(yx)^{n+1}(1+x) + 4b_n(y+(yx)^{n+1}) + (1+y)(Q_n-1)(yx)^n y(1+x) = 8b_n(yx)^{n+1}$.

This equation is equivalent to

$(yx)^{n+1}(1+x)+(1+y)(Q_n-1)(yx)^n y(1+x) = 4b_n y\{(xy)^n x - 1\}$.
But since $Q_n = yP_ny^{-1}$, it suffices to show

$$(xy)^n x(1+x)+(1+y)(P_n-1)(xy)^n(1+x) = 4b_n\{(xy)^n x-1\}. \tag{4.8}$$

Now take the image of both sides of (4.8) under f. Then,

$$\begin{aligned}
\text{LHS} &= \begin{bmatrix} 0 & b_n \\ c_n & 0 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix} + \begin{bmatrix} 2 & 0 \\ s_0 & 2 \end{bmatrix}\begin{bmatrix} \boldsymbol{a}-1 & \boldsymbol{b} \\ \boldsymbol{c} & \boldsymbol{d}-1 \end{bmatrix}\begin{bmatrix} 0 & 2b_n \\ 2c_n & d_n \end{bmatrix} \\
&= \begin{bmatrix} 4\boldsymbol{b}c_n & 4\boldsymbol{a}b_n - 2b_n + 2\boldsymbol{b}d_n \\ 4c_n(\boldsymbol{d}-1) & c_n + 2b_n s_0(\boldsymbol{a}-1) + 4b_n\boldsymbol{c} + d_n(2\boldsymbol{d}-3) \end{bmatrix}, \text{ and} \\
\text{RHS} &= 4b_n\begin{bmatrix} -1 & b_n \\ c_n & -1 \end{bmatrix}.
\end{aligned}$$

Therefore, we need to show

$$\begin{aligned}
&(i)\ 4\boldsymbol{b}c_n = -4b_n, \\
&(ii)\ 4\boldsymbol{a}b_n - 2b_n + 2\boldsymbol{b}d_n = 4b_n^2, \\
&(iii)\ 4c_n(\boldsymbol{d}-1) = 4b_nc_n, \\
&(iv)\ c_n + 2s_0b_n(\boldsymbol{a}-1) + 4b_n\boldsymbol{c} + d_n(2\boldsymbol{d}-3) = -4b_n.
\end{aligned} \tag{4.9}$$

First, (i) follows, since $4\boldsymbol{b}c_n = 4\boldsymbol{b}s_0b_n = -4b_n$, and (ii) follows, since $4\boldsymbol{a}b_n - 2b_n + 2\boldsymbol{b}d_n = 4b_n^2 - 2b_n - 2\boldsymbol{b}s_0b_n = 4b_n^2 - 2b_n + 2b_n = 4b_n^2$. Note that $d_n = a_{n-1} = -s_0b_n$. Also, (iii) follows, since $\boldsymbol{d} - 1 = b_n$. Finally, (iv) follows, since
$c_n + 2s_0b_n(\boldsymbol{a}-1) + 4b_n\boldsymbol{c} + d_n(2\boldsymbol{d}-3) = s_0b_n + 2s_0b_n(b_n-1) - 4b_n + + a_{n-1}(2\boldsymbol{a}-1) =$
$s_0b_n + 2s_0b_n^2 - 2s_0b_n - 4b_n - s_0b_n(2b_n-1) = -4b_n$.
This proves (3). □

5. Restatement of Theorem A.

Let $K(r)$ be an element of $H(p)$. Then $r = \frac{\beta}{\alpha}$ has a continued fraction expansion of the form: $\frac{\beta}{\alpha} = [pk_1, 2m_1, pk_2, 2m_2, \ldots, 2m_q, pk_{q+1}]$, where k_i, m_j are non-zero integers.
Using this form, we can construct a diagram of $K(r)$ as a 4-plat. First construct a 3-braid $\gamma = \sigma_2^{pk_1}\sigma_1^{2m_1}\sigma_2^{pk_2}\cdots\sigma_1^{2m_q}\sigma_2^{pk_{q+1}}$, where σ_i are Artin's generators of the 3-braid group. See Fig 5.1.

Figure 5.1: The Artin generators for 3-braids

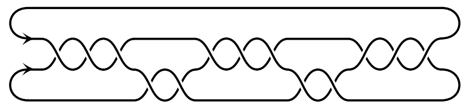

Figure 5.2: 2-bridge knot of $29/69 = [3, 2, 3, 2, -3]$

Close γ by joining the first and second strings (at the both ends) and then join the top and bottom of the third string by a simple arc as in Fig. 5.2. (For convenience, figures will be $\pi/2$ rotated.) We give downward orientation to the second and third strings.

Fig 5.2 shows the (oriented) 2-bridge knot obtained from the continued fraction $29/69 = [3, 2, 3, 2, -3]$. (A braid gives a knot diagram $D(r)$ of $K(r)$ if and only if $\sum_{j=1}^{q+1} k_j$ is odd.)

Remark 5.1. Although k_i and m_j are not 0, later in this paper, we need an appropriate interpretation of our continued fractions when some are 0. The following interpretations will be easily justified by checking their diagrams as 4-plats. If k_i $(i \neq 1, q+1)$ or m_j $(1 \leq j \leq q)$ is 0, then r is interpreted as
$[pk_1, 2m_1, \dots, pk_{i-1}, 2(m_{i-1} + m_i), pk_{i+1}, \dots, pk_{q+1}]$ or
$[pk_1, 2m_1, \dots, 2m_{j-1}, p(k_j + k_{j+1}), 2m_{j+1}, \dots, pk_{q+1}]$. If $k_1 = 0$ (or $k_{q+1} = 0$), then r is interpreted as $[pk_2, 2m_2, \dots, pk_{q+1}]$ (or $[pk_1, 2m_1, \dots, pk_q]$). Note that our continued fraction expansions start and end with pk.

Next we find a presentation of $G(K)$ from $D(r)$. Two (meridian) generators x and y are represented by loops that go around once under local maximal points from the left to the right as shown in Fig.5.3. The relation is obtained using x and y by a standard method. However, we describe this process more precisely.

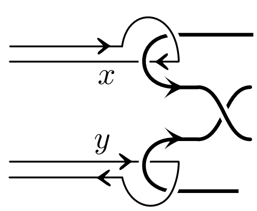

Figure 5.3: Generators for the knot group

First we divide $D(r)$ into $2q+3$ small pieces by $2q+2$ vertical lines L_j. See Fig 5.4. We define the elements $x_0, x_1, \cdots, x_{2q+1}, y_0, y_1, \cdots, y_{2q+1}$, $z_0, z_1, \cdots, z_{2q+1}$ in $F(x, y)$ as follows. Let Z_j, X_j, Y_j, respectively, be the points of intersection of L_j and the first, second and third strings. Then z_j, x_j, y_j, respectively, are represented

by loops that go around once under these points Z_j, X_j, Y_j from the left to the right. We note that $x_0 = x, y_0 = y$ and $z_0 = x$.

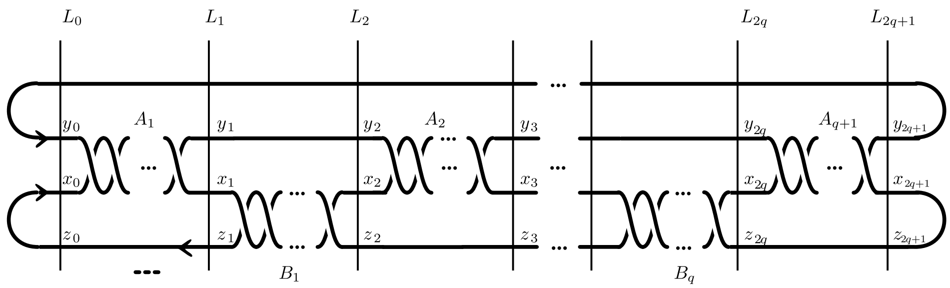

Figure 5.4: Elements in $F(x, y)$

By the standard method, x_j, y_j, z_j can be written as words of x and y. See Fig. 5.5. Let $A_{j+1} = y_{2j}x_{2j},\ j = 0, 1, \cdots, q$. Then x_{2j+1} and y_{2j+1} are given as follows:

(1) If $k_j = 2\ell_j,$ then $y_{2j+1} = A_{j+1}^{p\ell_j} y_{2j} A_{j+1}^{-p\ell_j},$ and $x_{2j+1} = A_{j+1}^{p\ell_j} x_{2j} A_{j+1}^{-p\ell_j}.$

(2) If $k_j = 2\ell_j + 1,$ then $y_{2j+1} = A_{j+1}^{p\ell_j+n+1} x_{2j} A_{j+1}^{-(p\ell_j+n+1)}$, and

$$x_{2j+1} = A_{j+1}^{p\ell_j+n} y_{2j} A_{j+1}^{-(p\ell_j+n)}.$$

(3) $y_{2j+2} = y_{2j+1}$ and $z_{2j+1} = z_{2j}, j = 0, 1, \cdots, q.$ (5.1)

Let $B_{j+1} = x_{2j+1}z_{2j+1}^{-1},\ j = 0, 1, \cdots, q-1$. Then x_{2j+2} and z_{2j+2} are given by

$$x_{2j+2} = B_{j+1}^{m_j} x_{2j+1} B_{j+1}^{-m_j}, \text{ and } z_{2j+2} = B_{j+1}^{m_j} z_{2j+1} B_{j+1}^{-m_j}. \tag{5.2}$$

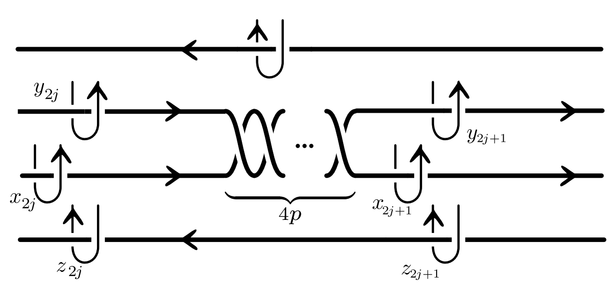

Figure 5.5: Rewriting process of letters

Then the relation of $G(K(r))$ is given by

$$y_{2q+1} = y, \text{ (orequivalently } x_{2q+1} = z_{2q+1}.) \tag{5.3}$$

Therefore, $G(K(r)) = \langle x, y | R \rangle$ is a Wirtinger presentation of $G(K(r))$, where $R = y_{2q+1}y^{-1}$. We note that relation (5.3) is a conjugate of the relation given by (2.2).

Now we can express the relation $y_{2q+1}y^{-1}$ as a product of conjugate of $R_0 = (xy)^n x(xy)^{-n}y^{-1}$:

$$y_{2q+1}y^{-1} = \prod_{j=1}^{m} u_j R_0^{\epsilon_j} u_j^{-1}, \text{ where } u_j \in F(x,y) \text{ and } \epsilon_j = \pm 1. \tag{5.4}$$

Let $\Phi_0 : \widetilde{A}(s_0) \to M_{2,2}(\mathbb{Z}[s_0])$ be a homomorphism defined by $\Phi_0 = \Phi|_{t=1}$, and hence $\Phi_0(x) = \rho(x)$ and $\Phi_0(y) = \rho(y)$. Then it follows from (3.4) that

$$\begin{aligned} &(1)\ \lambda_{\rho,K(r)}(1) = \det \left[\sum_{j=1}^{m} \epsilon_j u_j^{\Phi_0} \right], \text{ and} \\ &(2)\ \lambda_{\rho,K(r)}(-1) = \det \left[\sum_{j=1}^{m} (-1)^{\ell(u_j)} \epsilon_j u_j^{\Phi_0} \right], \end{aligned} \tag{5.5}$$

where $\ell(u_j)$ denotes the length of a word $u_j \in F(x,y)$.
Therefore, to prove Theorem A, it will be sufficient to show the following proposition.

Proposition 5.2. *(1) The element $\lambda(r) = \sum_{j=1}^{m} \epsilon_j u_j$ in the $\mathbb{Z}[s_0]$-algebra $\widetilde{A}(s_0)$ is a single element, namely, $\lambda(r) = \sum_{j=1}^{m} \epsilon_j u_j = \pm w$, for some element w in $F(x,y)$.*

(2) The element $\widetilde{\lambda}(r) = \sum_{j=1}^{m} (-1)^{\ell(u_j)} \epsilon_j u_j$ is a constant multiple of a single element, i.e. $\widetilde{\lambda}(r) = \pm \mu w$ for some $\mu \in \mathbb{Z}[s_0]$ and $w \in F(x,y)$.

6. Rewriting process

Now $R_0 = (xy)^n x(xy)^{-n}y^{-1}$ is a defining relation of $G(K(1/p))$. We denote $uR_0^{\epsilon}u^{-1}$ by $R_0^{\epsilon u}$, for $u \in F(x,y)$ and $\epsilon = \pm 1$.

In this section, we establish a rewriting process which transforms an element $w \in F(x,y)$ into the form $R_0^u w_0$, where $u \in \widetilde{A}(s_0)$ and $w_0 \in F(x,y)$. Since we are concerned on an element $\lambda(r)$ or $\widetilde{\lambda}(r)$ of $\widetilde{A}(s_0)$, we may write $R_0^u = R_0^{u'}$ if $u = u'$ in $\widetilde{A}(s_0)$, and $R_0^u R_0^v = R_0^{u+v} = R_0^v R_0^u$, where $u, v \in \widetilde{A}(s_0)$.

Lemma 6.1. *We have the following formulas involving R_0.*
(I) (1) $(yx)^{n+1}x(yx)^{-(n+1)} = R_0^y y$.
(2) $(yx)^n y(yx)^{-n} = R_0^{-1}x$.

(3) $(yx)^{2n+1}x(yx)^{-(2n+1)} = R_0^{(yx)^n y-1}x = R_0^{(xy)^n x-1}x.$
(4) $(yx)^{2n+1}y(yx)^{-(2n+1)} = R_0^{-(yx)^{n+1}+y}y.$
(5) $(yx)^{3n+2}x(yx)^{-(3n+2)} = R_0^{(yx)^p y-(yx)^{n+1}+y}y = R_0^{-(yx)^{n+1}}y.$
(6) $(yx)^{3n+1}y(yx)^{-(3n+1)} = R_0^{-(yx)^p+(yx)^n y-1}x = R_0^{(yx)^n y}x = R_0^{x(yx)^n}x.$
(II) For $k \geq 1$, $(R_0^g u)^k = R_0^{(1+u+\cdots+u^{k-1})g}u$*, where* $g \in \widetilde{A}(s_0)$ *and* $u \in F(x,y)$*.*

Proof. Since most of our proofs are straightforward, we prove only one of these formulas, say (I) (3). In fact, since $(yx)^n y = (xy)^n x$ in $\widetilde{A}(s_0)$, we have:

$$\begin{aligned}
(yx)^{2n+1}x(yx)^{-(2n+1)} &= (yx)^{2n+1}(y^{-1}x^{-1})^{2n}y^{-1} \\
&= (yx)^n y(xy)^n x(y^{-1}x^{-1})^n y^{-1}(x^{-1}y^{-1})^n \\
&= (yx)^n y R_0 (x^{-1}y^{-1})^n \\
&= R_0^{(yx)^n y}(xy)^n y(x^{-1}y^{-1})^n x^{-1}x \\
&= R_0^{(yx)^n y}R_0^{-1}x \\
&= R_0^{(yx)^n y-1}x \\
&= R_0^{(xy)^n x-1}x.
\end{aligned}$$

□

Lemma 6.2. *For the elements defined in Section 5, we have*
(1) For $j = 0,1,2,\cdots,2q+1$, $y_j x_j z_j = y$*, as elements of* $F(x,y)$*.*
(2) For $j = 0,1,2,\cdots,2q+1$*, we can write* $x_j = R_0^{v_j}x, y_j = R_0^{w_j}y$ *and* $z_j = R_0^{u_j}x$*, where* u_j, v_j, w_j *are elements of* $\widetilde{A}(s_0)$*.*

Proof. (1) is evident from the definition of x_j, y_j, z_j. Further for $j = 0$, (2) is evident. Consider the case $j = 1$.

If $k_1 = 2\ell_1$, we apply Lemma 6.1(I)(4) repeatedly to obtain $y_1 = (yx)^{p\ell_1}y(yx)^{-p\ell_1} = R^w y$ for some $w \in \widetilde{A}(s_0)$. If $k_1 = 2\ell_1+1$, then by Lemma 6.1(I)(3) we see that

$$\begin{aligned}
y_1 &= (yx)^{p\ell_1+n+1}x(yx)^{-(p\ell_1+n+1)} \\
&= (yx)^{n+1}(yx)^{p\ell_1}x(yx)^{-p\ell_1}(yx)^{(n+1)} \\
&= (yx)^{n+1}R_0^v x(yx)^{(n+1)} \quad \text{(for some } v \in \widetilde{A}(s_0)) \\
&= R_0^{(yx)^{n+1}v}(yx)^{n+1}x(yx)^{-(n+1)} \\
&= R_0^{(yx)^{n+1}v+y}y \quad \text{(by Lemma 6.1(I)(1)).)}
\end{aligned}$$

Also, $z_1 = z_0 = x = R_0^0 x$. Using Lemma 6.1(II), we can complete the proof by an easy inductive argument. The details are omitted. □

Note that to prove Theorem A we need more precise description of these elements w_j, u_j, v_j that will be given in Proposition 7.1.

Now Lemma 6.2 makes our proof of Theorem A considerably simpler as shown in the following proposition.

Proposition 6.3. *Let* $r = [pk_1, 2m_1, pk_2, 2m_2, \cdots, 2m_q, pk_{q+1}]$ *and* $r' = [pk'_1, 2m_1, pk'_2, 2m_2, \cdots, 2m_q, pk'_{q+1}]$. *Then* $\lambda(r) = \lambda(r')$ *if* $k_j \equiv k'$ *(mod 4) for* $j = 1, 2, \cdots, q+1$.

We should note that even though $\lambda(r) = \lambda(r')$, their twisted Alexander polynomials are different.

Proof of Proposition 6.3. Suppose that $k_j = 4$. By Lemma 6.2, we can write $y_{2j} = y_{2j-1} = R_0^w y$ and $x_{2j} = R_0^v x$ for some $w, v \in \widetilde{A}(s_0)$. Since $A_{j+1} = y_{2j}x_{2j} = R_0^w y R_0^v x = R_0^{w+yv}(yx)$, it follows that

$$\begin{aligned} y_{2j+1} &= A_{j+1}^{2p} y_{2j} A_{j+1}^{-2p} \\ &= (R_0^{w+yv} yx)^{2p} y_{2j} (R_0^{w+yv} yx)^{-2p}. \end{aligned}$$

Let $w + yv = g$. Then, by Lemma 6.1(II), we see

$$\begin{aligned} y_{2j+1} &= (R_0^g yx)^{2p} y_{2j} (x^{-1}y^{-1}R_0^{-g}) \\ &= R_0^{Q_{2p-1}g} (yx)^{2p} R_0^w y (x^{-1}y^{-1})^{2p} R_0^{-Q_{2p-1}g}. \end{aligned}$$

However, $Q_{2p-1} = 0$, since $(yx)^p = -1$, by Proposition 4.3 (2), and hence,

$$\begin{aligned} y_{2j+1} &= (yx)^{2p} R_0^w y (x^{-1}y^{-1})^{2p} \\ &= R_0^{(yx)^{2p}w} (yx)^{2p} y (x^{-1}y^{-1})^{2p} \\ &= R_0^w (yx)^{2p} y (x^{-1}y^{-1})^{2p}. \end{aligned}$$

By using Lemma 6.1(I)(4), we can show $(yx)^{2p}y(x^{-1}y^{-1})^{2p} = R_0^{(yx)^p y + y} y = y$, and hence, we have $y_{2j+1} = R_0^w y$. Similarly, we obtain $x_{2j+1} = R_0^v x$. The same argument works for $k_j = -4$.
Now we know that we may replace $\sigma_2^{pk_j}$ by $\sigma_2^{pk_j \pm 4p}$ in the braid presentation γ of $K(r)$ defined in Section 5 keeping $\lambda(r)$ unchanged. This proves Proposition 6.3. □

By Proposition 6.3, we may assume that

$$k_j = 1, 2 \text{ or } 3 \text{ for any } j = 1, 2, \cdots, q+1. \tag{6.1}$$

If $k_j \equiv 0$ (mod 4), then we may take $k_j = 0$ and r is reduced to a shorter continued fraction (Remark 5.1).

7. Proof of Theorem A (I), Proof of Proposition 5.2 (1)

Let $r = [pk_1, 2m_1, pk_2, 2m_2, \cdots, 2m_q, pk_{q+1}]$, and we may assume that $k_j = 1, 2$ or 3 for $1 \le j \le q+1$ and $m_j \ne 0$ for $1 \le j \le q$.

We want to show that

$$\lambda(r) = \sum_{j=1}^{m} \epsilon_j u_j = \pm w, \text{ for some } w \in F(x,y). \tag{7.1}$$

First we determine precisely the elements w_j, u_j, v_j.

Proposition 7.1. *For any $j = 0,1,2,\cdots,2q+1$, we can write $y_j = R_0^{w_j} y, x_j = R_0^{v_j} x$ and $z_j = R_0^{u_j} x$. Then we have the following;*
(1) $u_0 = u_1 = 0, v_0 = 0, w_0 = 0$.
(2) $u_0 = u_1, u_2 = u_3, \cdots, u_{2q} = u_{2q+1}$.
(3) $w_1 = w_2, w_3 = w_4, \cdots, w_{2q-1} = w_{2q}$.
(4) $u_{2j} = u_{2j+1} = \sum_{k=1}^{j} m_k(x-1)y^{-1}w_{2k-1}, j = 1,2,\cdots,q$.
(5) $v_j = u_j - y^{-1}w_j, j = 1,2,\cdots,2q+1$.
(6) For $j = 0,1,2,\cdots,q$,

$$w_{2j+1} = \begin{cases} 0 & \text{if } \sum_{i=1}^{j+1} k_i \equiv 0 \pmod 4 \\ y & \text{if } \sum_{i=1}^{j+1} k_i \equiv 1 \pmod 4 \\ y-(yx)^{n+1} & \text{if } \sum_{i=1}^{j+1} k_i \equiv 2 \pmod 4 \\ -(yx)^{n+1} & \text{if } \sum_{i=1}^{j+1} k_i \equiv 3 \pmod 4 \end{cases}$$

Since $\lambda(r) = w_{2q+1}$, it follows that if $K(r)$ is a knot, then $\lambda(r) = y$ or $-(yx)^{n+1}$. This proves (7.1) and hence Proposition 5.2 (1).

Proof of Proposition 7.1. Formulas (1)-(3) follow from the diagram $D(r)$. Further, (5) follows from Lemma 6.2(1). In fact, $y^{-1}y_jx_jz_j^{-1} = 1$ yields
$y^{-1}R_0^{w_j}yR_0^{v_j}xx^{-1}R_0^{-u_j} = R_0^{y^{-1}w_j}R_0^{v_j-u_j} = 1$ and hence $y^{-1}w_j + v_j - u_j = 0$ and $v_j = u_j - y^{-1}w_j$.

Now we prove (4) by induction on j. Since $B_1 = x_1z_1^{-1} = R_0^{v_1}xx^{-1} = R_0^{v_1}$, we see that

$$z_2 = B_1^{m_1}z_1B_1^{-m_1} = R_0^{m_1v_1}xR_0^{-m_1v_1} = R_0^{m_1(1-x)v_1}x. \tag{7.2}$$

Therefore, $u_2 = m_1(1-x)v_1$. Since $v_1 = u_1 - y^{-1}w_1 = -y^{-1}w_1$, we have $u_2 = m_1(x-1)y^{-1}w_1$. This proves $(4)_1$.

Next consider $u_{2\ell+2}$. By induction, we assume that
$u_{2\ell} = \sum_{j=1}^{\ell} m_j(x-1)y^{-1}w_{2j-1}$.
Since $B_{\ell+1} = x_{2\ell+1}z_{2\ell+1}^{-1} = R_0^{v_{2\ell+1}}xx^{-1}R_0^{-u_{2\ell}} = R_0^{v_{2\ell+1}-u_{2\ell}} = R_0^{-y^{-1}w_{2\ell+1}}$, we have

$$\begin{aligned} z_{2\ell+2} &= B_{\ell+1}^{m_{\ell+1}}z_{2\ell}B_{\ell+1}^{-m_{\ell+1}} \\ &= R_0^{-m_{\ell+1}y^{-1}w_{2\ell+1}}R_0^{u_{2\ell}}xR_0^{m_{\ell+1}y^{-1}w_{2\ell+1}} \\ &= R_0^{-m_{\ell+1}(1-x)y^{-1}w_{2\ell+1}+u_{2\ell}}x. \end{aligned} \tag{7.3}$$

Since $u_{2\ell+1} = u_{2\ell}$, we see

$$\begin{aligned} u_{2\ell+2} &= -m_{\ell+1}(1-x)y^{-1}w_{2\ell+1} + \sum_{j=1}^{\ell} m_j(x-1)y^{-1}w_{2j-1} \\ &= \sum_{j=1}^{\ell+1} m_j(x-1)y^{-1}w_{2j-1}. \end{aligned}$$

This proves $(4)_{\ell+1}$.

Finally we prove (6) by induction on j. Consider the initial case w_1.

Case 1. $k_1 = 1$. Since $A_1 = y_0x_0 = yx$, we see from Lemma 6.1(I)(1),

$$y_1 = A_1^{n+1}x_0A_1^{-(n+1)} = (yx)^{n+1}x(x^{-1}y^{-1})^{n+1} = R_0^y y. \tag{7.4}$$

Therefore, $w_1 = y$.

Case 2. $k_1 = 2$. As is seen in Case 1, we have from Lemma 6.1(I)(4)

$$y_1 = A_1^p y_0 A_1^{-p} = R_0^{-(yx)^{n+1}+y} y, \text{ and hence } w_1 = -(yx)^{n+1} + y. \tag{7.5}$$

Case 3. $k_1 = 3$. Then $pk_1 = 6n+3$ and

$$y_1 = A_1^{3n+2}x_0A_1^{-(3n+2)} = (yx)^{3n+2}x(x^{-1}y^{-1})^{3n+2} = R_0^{-(yx)^{n+1}} y, \tag{7.6}$$

and hence, $w_1 = -(yx)^{n+1}$. This proves $(6)_1$.

Next consider $w_{2\ell+1}$. Again the proof is divided into three cases: $k_{\ell+1} = 1,2,3$.

Case 1. $k_{\ell+1} = 1$ and $pk_{\ell+1} = 2n+1$.

Then, $A_{\ell+1} = y_{2\ell}x_{2\ell} = R_0^{w_{2\ell}} y R_0^{v_{2\ell}} x = R_0^{w_{2\ell}+yv_{2\ell}} yx$, and $y_{2\ell+1} = A_{\ell+1}^{n+1}x_{2\ell}A_{\ell+1}^{-(n+1)}$.

By induction assumption, we have:

$w_{2\ell} + yv_{2\ell} = w_{2\ell} + y(u_{2\ell} - y^{-1}w_{2\ell}) = yu_{2\ell}$. Therefore, $A_{\ell+1} = R_0^{yu_{2\ell}} yx$ and hence $y_{2\ell+1} = (R_0^{yu_{2\ell}}yx)^{n+1} R_0^{v_{2\ell}} x (R_0^{yu_{2\ell}}yx)^{-(n+1)}$.

Since $A_{\ell+1}^{n+1} = R_0^{Q_n yu_{2\ell}}(yx)^{n+1}$, it follows that

$$\begin{aligned} y_{2\ell+1} &= R_0^{Q_nyu_{2\ell}}(yx)^{n+1}R_0^{v_{2\ell}}x(yx)^{-(n+1)}R_0^{-Q_nyu_{2\ell}} \\ &= R_0^{Q_nyu_{2\ell}}R_0^{(yx)^{n+1}v_{2\ell}}(yx)^{n+1}x(yx)^{-(n+1)}R_0^{-Q_nyu_{2\ell}} \\ &= R_0^{Q_nyu_{2\ell}+(yx)^{n+1}v_{2\ell}+y}yR_0^{-Q_nyu_{2\ell}} \\ &= R_0^{(1-y)Q_nyu_{2\ell}+(yx)^{n+1}v_{2\ell}+y}y \end{aligned}$$

and hence

$$w_{2\ell+1} = (1-y)Q_nyu_{2\ell} + (yx)^{n+1}v_{2\ell} + y. \tag{7.7}$$

Since by (4), $u_{2\ell} = \sum_{j=1}^{\ell} m_j(x-1)y^{-1}w_{2j-1}$ and $v_{2\ell} = u_{2\ell} - y^{-1}w_{2\ell-1}$, we have

$$w_{2\ell+1} = \{(1-y)Q_ny + (yx)^{n+1}\}\left(\sum_{j=1}^{\ell} m_j(x-1)y^{-1}w_{2j-1}\right) - (yx)^{n+1}y^{-1}w_{2\ell-1} + y.$$

But $\{(1-y)Q_ny + (yx)^{n+1}\}(x-1) = 0$, by (4.5)(1), and hence,

$$w_{2\ell+1} = -(yx)^{n+1}y^{-1}w_{2\ell-1} + y. \tag{7.8}$$

Now, we consider the following four subcases separately.
Case (i) $\sum_{j=1}^{\ell} k_j \equiv 0 \pmod 4$, and thus $\sum_{j=1}^{\ell+1} k_j \equiv 1 \pmod 4$.
Then by induction $w_{2\ell-1} = 0$ and hence $w_{2\ell+1} = y$.
Case (ii) $\sum_{j=1}^{\ell} k_j \equiv 1 \pmod 4$, and thus $\sum_{j=1}^{\ell+1} k_j \equiv 2 \pmod 4$.
Then by induction $w_{2\ell-1} = y$ and hence $w_{2\ell+1} = -(yx)^{n+1} + y$.
Case (iii) $\sum_{j=1}^{\ell} k_j \equiv 2 \pmod 4$, and $\sum_{j=1}^{\ell+1} k_j \equiv 3 \pmod 4$.
Since $w_{2\ell-1} = y - (yx)^{n+1}$, we have

$$\begin{aligned} w_{2\ell+1} &= -(yx)^{n+1}y^{-1}(y - (yx)^{n+1}) + y \\ &= -(yx)^{n+1} + (yx)^{n+1}(yx)^n y + y \\ &= -(yx)^{n+1}. \end{aligned}$$

Case (iv) $\sum_{j=1}^{\ell} k_j \equiv 3 \pmod 4$, and $\sum_{j=1}^{\ell+1} k_j \equiv 0 \pmod 4$.
Since $w_{2\ell-1} = -(yx)^{n+1}$, we have

$$\begin{aligned} w_{2\ell+1} &= -(yx)^{n+1}y^{-1}(-(yx)^{n+1}) \ + \ y \\ &= (yx)^{n+1}(xy)^n x + y = (yx)^{n+1}(yx)^n y + y \\ &= -y + y \\ &= 0. \end{aligned}$$

This proves (6) for Case 1.

The same argument works for other cases.

Case 2. $k_{\ell+1} = 2$ and $pk_{\ell+1} = 4n + 2$. Then, $y_{2\ell+1} = A_{\ell+1}^p y_{2\ell} A_{\ell+1}^{-p}$.
Since $A_{\ell+1}^p = (R_0^{yu_{2\ell}} yx)^p$ and $y_{2\ell} = y_{2\ell-1} = R_0^{w_{2\ell-1}} y$, we have

$$y_{2\ell+1} = R_0^{Q_{p-1}yu_{2\ell}}(yx)^p R_0^{w_{2\ell-1}} y(yx)^{-p} R_0^{-Q_{p-1}yu_{2\ell}} = R_0^{\tau} y, \text{ where} \tag{7.9}$$

$$\tau = (1 - y)Q_{p-1}yu_{2\ell} + (yx)^p w_{2\ell-1} - (yx)^{n+1} + y. \tag{7.10}$$

Since by (4), $u_{2\ell} = \sum_{j=1}^{\ell} m_j(x - 1)y^{-1}w_{2j-1}$ and $(1 - y)Q_{p-1}y(x - 1) = 0$, by (4.5)(2), we have $\tau = (yx)^p w_{2\ell-1} - (yx)^{n+1} + y = -w_{2\ell-1} - (yx)^{n+1} + y$.

Again, we consider four subcases.
Case (i) $\sum_{j=1}^{\ell} k_j \equiv 0 \pmod 4$, and $\sum_{j=1}^{\ell+1} k_j \equiv 2 \pmod 4$.
Then by induction $w_{2\ell-1} = 0$ and hence $w_{2\ell+1} = -(yx)^{n+1} + y$.
Case (ii) $\sum_{j=1}^{\ell} k_j \equiv 1 \pmod 4$, and $\sum_{j=1}^{\ell+1} k_j \equiv 3 \pmod 4$.
Then by induction $w_{2\ell-1} = y$ and hence $w_{2\ell+1} = -(yx)^{n+1}$.
Case (iii) $\sum_{j=1}^{\ell} k_j \equiv 2 \pmod 4$, and $\sum_{j=1}^{\ell+1} k_j \equiv 0 \pmod 4$.
Since $w_{2\ell-1} = y - (yx)^{n+1}, w_{2\ell+1} = 0$.
Case (iv) $\sum_{j=1}^{\ell} k_j \equiv 3 \pmod 4$, and $\sum_{j=1}^{\ell+1} k_j \equiv 1 \pmod 4$.
Since $w_{2\ell-1} = -(yx)^{n+1}$, $w_{2\ell+1} = y$. This proves (6) for Case 2.

Case 3. $k_{\ell+1} = 3$ and $pk_{\ell+1} = 6n + 3$.
Then, $y_{2\ell+1} = A_{\ell+1}^{3n+2} x_{2\ell} A_{\ell+1}^{-(3n+2)}$. Since $A_{\ell+1}^{3n+2} = (R_0^{yu_{2\ell}} yx)^{3n+2}$ $= R_0^{Q_{3n+1}yu_{2\ell}}(yx)^{3n+2}$, it follows that $y_{2\ell+1} = R_0^{\tau} y$, where

$$\tau = \{(1 - y)Q_{3n+1}y + (yx)^{3n+2}\}u_{2\ell} - (yx)^{3n+2}y^{-1}w_{2\ell-1} - (yx)^{n+1}. \tag{7.11}$$

Since $\{(1-y)Q_{3n+1}y+(yx)^{3n+2}\}(x-1)=0$ by (4.5)(3), we have
$\tau=-(yx)^{3n+2}y^{-1}w_{2\ell-1}-(yx)^{n+1}=(yx)^{n+1}y^{-1}w_{2\ell-1}-(yx)^{n+1}$.
Case (i) $\sum_{j=1}^{\ell}k_j\equiv 0 \pmod 4$, and $\sum_{j=1}^{\ell+1}k_j\equiv 3 \pmod 4$.
Then by induction $w_{2\ell-1}=0$ and $w_{2\ell+1}=-(yx)^{n+1}$.
Case (ii) $\sum_{j=1}^{\ell}k_j\equiv 1 \pmod 4$, and $\sum_{j=1}^{\ell+1}k_j\equiv 0 \pmod 4$.
Then $w_{2\ell-1}=y$ and hence $w_{2\ell+1}=0$.
Case (iii) $\sum_{j=1}^{\ell}k_j\equiv 2 \pmod 4$, and $\sum_{j=1}^{\ell+1}k_j\equiv 1 \pmod 4$.
Since $w_{2\ell-1}=y-(yx)^{n+1}$, we have
$w_{2\ell+1}=(yx)^{n+1}-(yx)^{n+1}(xy)^nx-(yx)^{n+1}=y$.
Case (iv) $\sum_{j=1}^{\ell}k_j\equiv 3 \pmod 4$, and $\sum_{j=1}^{\ell+1}k_j\equiv 2 \pmod 4$.
Since $w_{2\ell-1}=-(yx)^{n+1}$, $w_{2\ell+1}=y-(yx)^{n+1}$.
This proves (6) for Case 3, and the proof of the first part of Theorem A is complete.

8. Proof of Theorem A. (II), Proof of Proposition 5.2(2)

In this section, we prove that if $K(r)$ is a knot, then

$$\widetilde{\lambda}(r)=\sum_{j=1}^{m}(-1)^{\ell(u_j)}\epsilon_ju_j=\pm\mu w,\ \text{where } w\in F(x,y) \text{ and } \mu\in\mathbb{Z}[s_0]. \tag{8.1}$$

For simplicity, to each element u in $\widetilde{A}(s_0)$, say $u=\sum_j\epsilon_ju_j$, we write $\widetilde{u}=\sum_j(-1)^{\ell(u_j)}\epsilon_ju_j$.

First we notice a similar proposition to Proposition 6.3 holds. Since a proof is exactly the same, we omit the details.

Proposition 8.1. *Let* $r=[pk_1,2m_1,pk_2,2m_2,\cdots,2m_q,pk_{q+1}]$ *and* $r'=[pk_1',2m_1,pk_2',2m_2,\cdots,2m_q,pk_{q+1}']$. *Then* $\widetilde{\lambda}(r)=\widetilde{\lambda}(r')$ *if* $k\equiv k'$ *(mod 4) for* $j=1,2,\cdots,2q+1$.

To evaluate $\widetilde{w}_j,\widetilde{u}_j$ and $\widetilde{v}_j$, we repeat the same argument that was used in Section 7. But we employ Lemma 4.6 instead of Lemma 4.5.

Proposition 8.2. *Let* $r=[pk_1,2m_1,pk_2,2m_2,\cdots,2m_q,pk_{q+1}]$, *where* $k_j=1,2,$

or 3 *for any* $j \geq 1$. *Then we have*

(1) $\widetilde{w}_0 = 0, \widetilde{u}_0 = 0$ *and* $\widetilde{v}_0 = 0$.

(2) *For* $j \geq 1, \widetilde{w}_{2j-1} = \widetilde{w}_{2j}$, *and* $\widetilde{u}_{2j-2} = \widetilde{u}_{2j-1}$.

(3) *For any* $j \geq 0, \widetilde{v}_j = \widetilde{u}_j + y^{-1}\widetilde{w}_j$.

(4) *Suppose* $q \geq 0$.

(*a*) *If* $k_{q+1} = 1$, *then*
$\widetilde{w}_{2q+1} = \{-(1+y)Q_n y + (yx)^{n+1}\}\widetilde{u}_{2q} + (yx)^{n+1}y^{-1}\widetilde{w}_{2q} - y$.

(*b*) *If* $k_{q+1} = 2$, *then*
$\widetilde{w}_{2q+1} = \{-(1+y)Q_{2n}y\}\widetilde{u}_{2q} - \widetilde{w}_{2q} - (yx)^{n+1} - y$.

(*c*) *If* $k_{q+1} = 3$, *then*
$\widetilde{w}_{2q+1} = \{-(1+y)Q_{3n+1}y - (yx)^{n+1}\}\widetilde{u}_{2q} - (yx)^{n+1}y^{-1}\widetilde{w}_{2q} - (yx)^{n+1}$.

(5) *For any* $j \geq 1, \widetilde{u}_{2j} = m_j(x+1)y^{-1}\widetilde{w}_{2j-1} + \widetilde{u}_{2j-1}$. (8.2)

Proof. (1) and (2) follow immediately, noting $\widetilde{x} = -x$. Also (3) follows from Proposition 7.1(5), since $\widetilde{y} = -y$. Next we prove (5). Consider $B_j = x_{2j-1}z_{2j-1}^{-1} = R_0^{v_{2j-1}-u_{2j-1}} = R_0^{-y^{-1}w_{2j-1}}$. Then by (7.3), we see $z_{2j} = B_j^{m_j} z_{2j-1} B_j^{-m_j} = R_0^{-m_j(1-x)y^{-1}w_{2j-1}+u_{2j-1}}x$, and hence, $u_{2j} = -m_j(1-x)y^{-1}w_{2j-1} + u_{2j-1}$ and $\widetilde{u}_{2j} = m_j(1+x)y^{-1}\widetilde{w}_{2j-1} + \widetilde{u}_{2j-1}$.

Finally, we prove (4) by induction. For the initial case $q = 0$, Proposition 8.2 holds. In fact, if $k_1 = 1$, (7.4) shows that $w_1 = y$ and $\widetilde{w}_1 = -y$. If $k_1 = 2$, then from (7.5) we see that $w_1 = -(yx)^{n+1} + y$ and $\widetilde{w}_1 = -(yx)^{n+1} - y$. If $k_1 = 3$, then $w_1 = -(yx)^{n+1} = \widetilde{w}_1$ by (7.6).

Now suppose Proposition 8.2(4) holds for q and prove it for $q+1$.
If $k_{q+1} = 1$, then (7.7) yields, since $v_{2q} = u_{2q} - y^{-1}w_{2q}$, $w_{2q+1} = (1-y)Q_n y u_{2q} + (yx)^{n+1}(u_{2q} - y^{-1}w_{2q}) + y$.
By taking a tilde on each element in both sides, we obtain (4)(a). If $k_{q+1} = 2$, then since $(yx)^{2n+1} = -1$, $w_{2q+1} = (1-y)Q_{2n}yu_{2q} - w_{2q} - (yx)^{n+1} + y$ by (7.10). By taking a tilde on each element, we have (4)(b).
If $k_{q+1} = 3$, then (7.11) yields
$w_{2q+1} = \{(1-y)Q_{3n+1}y + (yx)^{3n+2}\}u_{2q} - (yx)^{3n+2}y^{-1}w_{2q} - (yx)^{n+1}$ and since $(yx)^{2n+1} = 1$, (4)(c) follows by taking a tilde on each element. This proves Proposition 8.2. □

Theorem 8.3. *Let* $r = [pk_1, 2m_1, pk_2, 2m_2, \cdots, 2m_q, pk_{q+1}]$,
$r' = [pk_1, 2m_1, pk_2, 2m_2, \cdots, 2m_{q-1}, pk_q]$, *and*
$\widehat{r} = [pk_1, 2m_1, pk_2, 2m_2, \cdots, 2m_{q-1}, p(k_q + k_{q+1})]$.
Then $\widetilde{\lambda}(r)$ *is of the form:*

(1) If $\sum_{j=1}^{q+1} k_j \equiv 0 \pmod 4$, *then* $\widetilde{\lambda}(r) = F_0(r)(y - (yx)^{n+1})$.

(2) If $\sum_{j=1}^{q+1} k_j \equiv 1 \pmod 4$, *then* $\widetilde{\lambda}(r) = F_1(r)y$.

(3) If $\sum_{j=1}^{q+1} k_j \equiv 2 \pmod 4$*, then* $\widetilde{\lambda}(r) = F_2(r)(y + (yx)^{n+1})$.

(4) If $\sum_{j=1}^{q+1} k_j \equiv 3 \pmod 4$*, then* $\widetilde{\lambda}(r) = F_3(r)(yx)^{n+1}$. *(8.3)*

Here $F_j(r), 0 \le j \le 3$*, are complex numbers in* $\mathbb{Z}[s_0]$*, and these numbers are determined inductively as follows.*

(I) $F_0([0]) = 0, F_1([p]) = F_2([2p]) = F_3([3p]) = -1$

(II) Suppose $q > 0$.

$$
\begin{aligned}
&(1)\ \textit{If}\ \ k_{q+1} = 1,\ \ \textit{then}\\
&\quad (i)\ F_0(r) = 4m_q b_n F_3(r') + F_0(\widehat{r})\\
&\quad (ii)\ F_1(r) = -8m_q b_n F_0(r') + F_1(\widehat{r})\\
&\quad (iii)\ F_2(r) = -4m_q b_n F_1(r') + F_2(\widehat{r})\\
&\quad (iv)\ F_3(r) = -8m_q b_n F_2(r') + F_3(\widehat{r})\\
\\
&(2)\ \textit{If}\ \ k_{q+1} = 2,\ \ \textit{then}\\
&\quad (i)\ F_0(r) = 8m_q b_n F_2(r') + F_0(\widehat{r})\\
&\quad (ii)\ F_1(r) = 8m_q b_n F_3(r') + F_1(\widehat{r})\\
&\quad (iii)\ F_2(r) = -8m_q b_n F_0(r') + F_2(\widehat{r})\\
&\quad (iv)\ F_3(r) = -8m_q b_n F_1(r') + F_3(\widehat{r})\\
\\
&(3)\ \textit{If}\ \ k_{q+1} = 3,\ \ \textit{then}\\
&\quad (i)\ F_0(r) = 4m_q b_n F_1(r') + F_0(\widehat{r})\\
&\quad (ii)\ F_1(r) = 8m_q b_n F_2(r') + F_1(\widehat{r})\\
&\quad (iii)\ F_2(r) = 4m_q b_n F_3(r') + F_2(\widehat{r})\\
&\quad (iv)\ F_3(r) = -8m_q b_n F_0(r') + F_3(\widehat{r})
\end{aligned}
\tag{8.4}
$$

Here b_n *is the* $(1,2)$ *entry of the matrix* $(XY)^n$*, see (4.1).*

Remark 8.4. We use these formulas as follows. For example, suppose $k_{q+1} = 1$. If $\sum_{j=1}^{q+1} k_j \equiv 0 \pmod 4$, then we see by (8.3)(1), $\widetilde{\lambda}(r) = F_0(r)(y - (yx)^{n+1})$. In this case, since $k_{q+1} = 1$, it follows $\sum_{j=1}^{q} k_j \equiv 3 \pmod 4$ and hence by (8.3)(4), we see $\widetilde{\lambda}(r') = F_3(r')(yx)^{n+1}$. Further, $\sum_{j=1}^{q-1} k_j + (k_q + k_{q+1}) \equiv 0 \pmod 4$ implies that $\widetilde{\lambda}(\widehat{r}) = F_0(\widehat{r})(y - (yx)^{n+1})$. We know inductively $F_3(r')$ and $F_0(\widehat{r})$, since the lengths of r' and $\widehat{r}$ are shorter than that of r, and therefore, $F_0(r)$ is determined by (8.4)(II) (1)(i) using $F_3(r')$ and $F_0(\widehat{r})$. We list $F_j(r)$ for $q = 1$ in the next section.

Proof of Theorem 8.3. We use induction on q. For the initial case, $q = 0$, since $r = [pk_1]$ and $r' = \widehat{r} = [0]$, (8.3) and (8.4)(I) follow from (8.2)(4). Note $\widetilde{u}_0 = \widetilde{w}_0 = 0$.

Next consider the case $q = 1$, i.e. $r = [pk_1, 2m_1, pk_2]$. Then
$r' = [pk_1]$ and $\widehat{r} = [p(k_1 + k_2)]$.

Case (1) $k_2 = 1$.

By (8.2)(4)(a), we have $\widetilde{w}_3 = [-(1 + y)Q_n y + (yx)^{n+1}]\widetilde{u}_2 + (yx)^{n+1} y^{-1} \widetilde{w}_2 - y$.

Since $\widetilde{u}_2 = m_1(x+1)y^{-1}\widetilde{w}_1$ and $\widetilde{w}_2 = \widetilde{w}_1$, we see
$\widetilde{w}_3 = [-(1+y)Q_n y + (yx)^{n+1}]m_1(1+x)y^{-1}\widetilde{w}_1 + (yx)^{n+1}y^{-1}\widetilde{w}_1 - y$.
Further by (4.6)(1), we have

$$\widetilde{w}_3 = -4b_n m_1(y + (yx)^{n+1})y^{-1}\widetilde{w}_1 + (yx)^{n+1}y^{-1}\widetilde{w}_1 - y. \tag{8.5}$$

Now we apply (8.3).
If $k_1 = 1$, then $\widetilde{w}_1 = -y$, and hence $\widetilde{w}_3 = (4b_n m_1 - 1)(y + (yx)^{n+1})$. This proves (8.3) for this case.
If $k_1 = 2$, then $\widetilde{w}_1 = -(y + (yx)^{n+1})$, and hence

$$\begin{aligned}\widetilde{w}_3 &= 4b_n m_1(y + (yx)^{n+1})y^{-1}(y + (yx)^{n+1}) - (yx)^{n+1}y^{-1}(y + (yx)^{n+1}) - y\\ &= (8b_n m_1 - 1)(yx)^{n+1}.\end{aligned}$$

If $k_1 = 3$, then $\widetilde{w}_1 = -(yx)^{n+1}$, and hence

$$\begin{aligned}\widetilde{w}_3 &= 4b_n m_1(y + (yx)^{n+1})y^{-1}(yx)^{n+1} - (yx)^{n+1}y^{-1}(yx)^{n+1} - y\\ &= 4b_n m_1((yx)^{n+1} - y).\end{aligned}$$

This proves (8.3) for Case (1), $k_2 = 1$.

Since similar arguments work for other cases, we skip details.

Case (2) $k_2 = 2$.
By (8.2)(4)(b), we have $\widetilde{w}_3 = \{-(1+y)Q_{2n}y\}\widetilde{u}_2 - \widetilde{w}_2 - (yx)^{n+1} - y$.
By (4.6)(3), it becomes to

$$\widetilde{w}_3 = -8b_n m_1(yx)^{n+1}y^{-1}\widetilde{w}_1 - \widetilde{w}_1 - (yx)^{n+1} - y. \tag{8.6}$$

As before, compute $\widetilde{w}_3$ to each case $k_1 = 1, 2$ or 3 to prove (8.3).

Case (3) $k_2 = 3$.
By (8.2)(4)(c), we see
$\widetilde{w}_3 = \{-(1+y)Q_{3n+1}y - (yx)^{n+1}\}\widetilde{u}_2 - (yx)^{n+1}y^{-1}\widetilde{w}_2 - (yx)^{n+1}$.
By (4.6)(5), it becomes to

$$\widetilde{w}_3 = 4b_n m_1(y - (yx)^{n+1})y^{-1}\widetilde{w}_1 - (yx)^{n+1}y^{-1}\widetilde{w}_1 - (yx)^{n+1}. \tag{8.7}$$

Computation of $\widetilde{w}_3$ to each case $k_1 = 1, 2$ or 3 completes the proof for $q = 1$.

Next we assume that Theorem 8.3 holds for any r with length less than $2q+1$. First consider the case where $k_{q+1} = 1$. We divide our proof into three subcases.

Case (1.1) $(k_q, k_{q+1}) = (1, 1)$. From (8.2)(4)(a), we have
$\widetilde{w}_{2q+1} = \{-(1+y)Q_n y + (yx)^{n+1}\}\widetilde{u}_{2q} + (yx)^{n+1}y^{-1}\widetilde{w}_{2q} - y$.
Since $\widetilde{u}_{2q} = m_q(1+x)y^{-1}\widetilde{w}_{2q-1} + \widetilde{u}_{2q-1}$, it follows that

$$\begin{aligned}\widetilde{w}_{2q+1} &= \{-(1+y)Q_n y + (yx)^{n+1}\}(m_q(1+x)y^{-1}\widetilde{w}_{2q-1} + \widetilde{u}_{2q-1})\\ &\quad + (yx)^{n+1}y^{-1}\widetilde{w}_{2q} \ - \ y\\ &= \{-(1+y)Q_n y + (yx)^{n+1}\}m_q(1+x)y^{-1}\widetilde{w}_{2q-1}\\ &\quad + \{-(1+y)Q_n y + (yx)^{n+1}\}\widetilde{u}_{2q-2}\\ &\quad + (yx)^{n+1}y^{-1}\widetilde{w}_{2q-1} - y,\end{aligned}$$

since $\widetilde{w}_{2q} = \widetilde{w}_{2q-1}$ and $\widetilde{u}_{2q-1} = \widetilde{u}_{2q-2}$.

Let $A = \{-(1+y)Q_n y + (yx)^{n+1}\}(1+x)$ and
$B = \{-(1+y)Q_n y + (yx)^{n+1}\}\widetilde{u}_{2q-2} + (yx)^{n+1}y^{-1}\widetilde{w}_{2q-1} - y$.
Then $\widetilde{w}_{2q+1} = Am_q y^{-1}\widetilde{w}_{2q-1} + B$.

First we claim that $B = \widetilde{\lambda}(\widehat{r})$. To prove this claim we should note that, since $k_q = 1$, by induction assumption,
$\widetilde{w}_{2q-1} = \{-(1+y)Q_n y + (yx)^{n+1}\}\widetilde{u}_{2q-2} + (yx)^{n+1}y^{-1}\widetilde{w}_{2q-2} - y$. Therefore,

$$\begin{aligned}
B &= \{-(1+y)Q_n y + (yx)^{n+1}\}\widetilde{u}_{2q-2} \\
&\quad + (yx)^{n+1}y^{-1}\Big[\{-(1+y)Q_n y + (yx)^{n+1}\}\widetilde{u}_{2q-2} + (yx)^{n+1}y^{-1}\widetilde{w}_{2q-2} - y\Big] - y \\
&= \Big[-(1+y)Q_n y + (yx)^{n+1} + (yx)^{n+1}y^{-1}\big\{-(1+y)Q_n y + (yx)^{n+1}\big\}\Big]\widetilde{u}_{2q-2} \\
&\quad + (yx)^{n+1}y^{-1}\{(yx)^{n+1}y^{-1}\widetilde{w}_{2q-2} - y\} - y.
\end{aligned}$$

Since $k_q + k_{q+1} = 2$, it suffices to show, using (8,2)(4)(b),

$$\begin{aligned}
&(i)\ -(1+y)Q_n y + (yx)^{n+1} - (yx)^{n+1}y^{-1}(1+y)Q_n y + (yx)^{n+1}y^{-1}(yx)^{n+1} \\
&\qquad = -(1+y)Q_{2n}y, \text{ and} \\
&(ii)\ (yx)^{n+1}y^{-1}(yx)^{n+1}y^{-1} = -1.
\end{aligned} \tag{8.8}$$

Proof of (8.8). (ii) follows immediately, and then, (i) becomes to
$-(1+y)Q_n y + (yx)^{n+1} - (yx)^{n+1}y^{-1}(1+y)Q_n y - y = -(1+y)Q_{2n}y$.
Since $(yx)^{n+1} = (yx)(yx)^n = y(yx)^n y$, the above equation is equivalent to
(i)' $-(1+y)Q_n + y(yx)^n - y(yx)^n(1+y)Q_n - 1 = -(1+y)Q_{2n}$.
Since $Q_{2n} = Q_n + (yx)^n(Q_n - 1)$, we see
LHS (of (i)')
$= -(1+y)Q_{2n} + (1+y)(yx)^n(Q_n - 1) + y(yx)^n - y(yx)^n(1+y)Q_n - 1$
$= -(1+y)Q_{2n} + (1+y)(yx)^nQ_n - (1+y)(yx)^n + y(yx)^n - y(yx)^n(1+y)Q_n - 1$
$= -(1+y)Q_{2n} + \Big\{(1+y)(yx)^n - y(yx)^n(1+y)\Big\}Q_n - (yx)^n - 1$
$= -(1+y)Q_{2n} + (yx)^n(1-yx)Q_n - (yx)^n - 1$
$= -(1+y)Q_{2n} + (yx)^n(1-(yx)^{n+1}) - (yx)^n - 1$
$= -(1+y)Q_{2n} - (yx)^{2n+1} - 1$
$= -(1+y)Q_{2n}$.
This proves (8.8) and $B = \widetilde{\lambda}(\widehat{r})$. Therefore, we have

$$\widetilde{w}_{2q+1} = Am_q y^{-1}\widetilde{w}_{2q-1} + \widetilde{\lambda}(\widehat{r}). \tag{8.9}$$

We note (4.6)(1) shows us that $A = -4b_n(y + (yx)^{n+1})$.

To prove (8.4)(II)(1), we consider the following four cases separately.

Case (i) Suppose $\sum_{j=1}^{q+1} k_j \equiv 0 \pmod 4$. Since $k_{q+1} = 1, \sum_{j=1}^{q} k_j \equiv 3 \pmod 4$ and hence, by induction assumption, $\widetilde{w}_{2q-1} = F_3(r')(yx)^{n+1}$. Therefore

$$\begin{aligned}
Am_q y^{-1}\widetilde{w}_{2q-1} &= m_q F_3(r')\Big\{-4b_n(y + (yx)^{n+1})\Big\}y^{-1}(yx)^{n+1} \\
&= -4m_q F_3(r')b_n((yx)^{n+1} - y).
\end{aligned}$$

Also, by induction, $\widetilde{\lambda}(\widehat{r}) = F_0(\widehat{r})(y - (yx)^{n+1})$, and hence
$\widetilde{\lambda}(r) = (4m_q F_3(r')b_n + F_0(\widehat{r}))(y - (yx)^{n+1})$. This proves (8.4)(II)(1)(i).

Case (ii) $\sum_{j=1}^{q+1} k_j \equiv 1 \pmod 4$. Then $\sum_{j=1}^{q} k_j \equiv 0 \pmod 4$ and hence, by induction assumption, we obtain that $\widetilde{w}_{2q-1} = F_0(r')(y - (yx)^{n+1})$, and therefore,

$$\begin{aligned}
Am_q y^{-1}\widetilde{w}_{2q-1} &= -4m_q F_0(r')b_n(y + (yx)^{n+1})y^{-1}(y - (yx)^{n+1})\\
&= -4m_q F_0(r')b_n y(1 + (xy)^n x)(1 - (xy)^n x)\\
&= -4m_q F_0(r')b_n y(1 - (xy)^n xy(xy)^n)\\
&= -8m_q F_0(r')b_n y.
\end{aligned}$$

Also, by induction, $\widetilde{\lambda}(\widehat{r}) = F_1(\widehat{r})y$, and hence $\widetilde{\lambda}(r) = (-8m_q F_0(r')b_n + F_1(\widehat{r}))y$.

Case (iii). $\sum_{j=1}^{q+1} k_j \equiv 2 \pmod 4$. Then $\sum_{j=1}^{q} k_j \equiv 1 \pmod 4$ and hence, by induction assumption, $\widetilde{\lambda}(r') = F_1(r')y$, and

$$\begin{aligned}
Am_q y^{-1}\widetilde{w}_{2q-1} &= -4m_q F_1(r')b_n(y + (yx)^{n+1})y^{-1}y\\
&= -4m_q F_1(r')b_n(y + (yx)^{n+1}).
\end{aligned}$$

On the other hand, $\widetilde{\lambda}(\widehat{r}) = F_2(\widehat{r})(y + (yx)^{n+1})$, and hence
$\widetilde{\lambda}(r) = (-4m_q F_1(r')b_n + F_2(\widehat{r}))(y + (yx)^{n+1})$.

Case (iv) $\sum_{j=1}^{q+1} k_j \equiv 3 \pmod 4$. Then $\sum_{j=1}^{q} k_j \equiv 2 \pmod 4$ and hence, $\widetilde{\lambda}(r') = F_2(r')(y + (yx)^{n+1})$, and

$$\begin{aligned}
Am_q y^{-1}\widetilde{w}_{2q-1} &= -4m_q F_2(r')b_n(y + (yx)^{n+1})y^{-1}(y + (yx)^{n+1})\\
&= -4m_q F_2(r')b_n y(1 + (xy)^n x)(1 + (xy)^n x)\\
&= -4m_q F_2(r')b_n y(1 + 2(xy)^n x + (xy)^{2n+1})\\
&= -8m_q F_2(r')b_n (yx)^{n+1}.
\end{aligned}$$

Also, by induction, $\widetilde{\lambda}(\widehat{r}) = F_3(\widehat{r})(yx)^{n+1}$, and hence
$\widetilde{\lambda}(r) = (-8m_q F_2(r')b_n + F_3(\widehat{r}))(yx)^{n+1}$.
Therefore, Theorem 8.3 is proved for this case.

For other cases, we use essentially the same argument, although calculations for some cases are a bit complicated. We just state the final forms and details will be omitted.

Case (2.1) $(k_q, k_{q+1}) = (2, 1)$

First we write $\widetilde{w}_{2q+1} = Am_q y^{-1}\widetilde{w}_{2q-1} + B$, where
$A = \{-(1+y)Q_n y + (yx)^{n+1}\}(1+x)$ and
$B = \{-(1+y)Q_n y + (yx)^{n+1} - (yx)^{n+1}y^{-1}(1+y)Q_{2n}y\}\widetilde{u}_{2q-2}$
$\quad + (yx)^{n+1}y^{-1}(-\widetilde{w}_{2q-2} - (yx)^{n+1} - y) - y$.
Then, we can show that

$$\begin{aligned}
&(1)\ -(1+y)Q_n y + (yx)^{n+1} - (yx)^{n+1}y^{-1}(1+y)Q_{2n}y\\
&\quad = -(1+y)Q_{3n+1}y - (yx)^{n+1},\ \text{and}\\
&(2)\ (yx)^{n+1}y^{-1}(-\widetilde{w}_{2q-2} - (yx)^{n+1} - y) - y\\
&\quad = -(yx)^{n+1}y^{-1}\widetilde{w}_{2q-2} - (yx)^{n+1}.
\end{aligned} \tag{8.10}$$

Therefore, we see that $B = \widetilde{\lambda}(\widehat{r})$, and further, $A = -4b_n(y + (yx)^{n+1})$, and thus, $\widetilde{w}_{2q+1} = -4m_q b_n(y + (yx)^{n+1})y^{-1}\widetilde{w}_{2q-1} + \widetilde{\lambda}(\widehat{r})$.

Case (i) Suppose $\sum_{j=1}^{q+1} k_j \equiv 0 \pmod 4$.
Then $\sum_{j=1}^{q} k_j \equiv 3 \pmod 4$ and hence, by induction assumption, $\widetilde{w}_{2q-1} = F_3(r')(yx)^{n+1}$, and

$$\begin{aligned} Am_q y^{-1}\widetilde{w}_{2q-1} &= -4m_q F_3(r')b_n(y + (yx)^{n+1})y^{-1}(yx)^{n+1} \\ &= -4m_q F_3(r')b_n((yx)^{n+1} - y). \end{aligned}$$

Therefore, $\widetilde{\lambda}(r) = (4m_q F_3(r')b_n + F_0(\widehat{r}))(y - (yx)^{n+1})$.
Case (ii). $\sum_{j=1}^{q+1} k_j \equiv 1 \pmod 4$.
Then $\sum_{j=1}^{q} k_j \equiv 0 \pmod 4$ and hence, $\widetilde{w}_{2q-1} = F_0(r')(y - (yx)^{n+1})$, and

$$\begin{aligned} Am_q y^{-1}\widetilde{w}_{2q-1} &= -4m_q F_0(r')b_n(y + (yx)^{n+1})y^{-1}(y - (yx)^{n+1}) \\ &= -8m_q F_0(r')b_n y. \end{aligned}$$

Also by induction, $\widetilde{\lambda}(\widehat{r}) = F_1(\widehat{r})y$, and hence $\widetilde{\lambda}(r) = (-8m_q F_0(r')b_n + F_1(\widehat{r}))y$.
Case (iii) $\sum_{j=1}^{q+1} k_j \equiv 2 \pmod 4$.
Then $\sum_{j=1}^{q} k_j \equiv 1 \pmod 4$ and, by induction assumption, $\widetilde{w}_{2q-1} = F_1(r')y$. Therefore, $Am_q y^{-1}\widetilde{w}_{2q-1} = -4m_q F_1(r')b_n(y + (yx)^{n+1})y^{-1}y$.
On the other hand, $\widetilde{\lambda}(\widehat{r}) = F_2(\widehat{r})(y + (yx)^{n+1})$, and hence
$\widetilde{\lambda}(r) = (-4m_q F_1(r')b_n + F_2(\widehat{r}))(y + (yx)^{n+1})$.
Case (iv) $\sum_{j=1}^{q+1} k_j \equiv 3 \pmod 4$.
Then $\sum_{j=1}^{q} k_j \equiv 2 \pmod 4$ and, by induction assumption, $\widetilde{w}_{2q-1} = F_2(r')(y + (yx)^{n+1})$. Therefore,

$$\begin{aligned} Am_q y^{-1}\widetilde{w}_{2q-1} &= -4m_q F_2(r')b_n(y + (yx)^{n+1})y^{-1}(y + (yx)^{n+1}) \\ &= -8m_q F_2(r')b_n(yx)^{n+1}. \end{aligned}$$

Since $\widetilde{\lambda}(\widehat{r}) = F_3(\widehat{r})(yx)^{n+1}$, we have $\widetilde{\lambda}(r) = (-8m_q F_2(r')b_n + F_3(\widehat{r}))(yx)^{n+1}$. Therefore, for this case, Theorem 8.3 is proved.
Case (3.1) $(k_q, k_{q+1}) = (3, 1)$
As above, we write $\widetilde{w}_{2q+1} = Am_q y^{-1}\widetilde{w}_{2q-1} + B$, where
$A = \{-(1+y)Q_n y + (yx)^{n+1}\}(1+x) = -4b_n(y + (yx)^{n+1})$, and
$B = \{-(1+y)Q_n y + (yx)^{n+1} + (yx)^{n+1}y^{-1}(-(1+y)Q_{3n+1}y - (yx)^{n+1})\}\widetilde{u}_{2q-2}$
$+ (yx)^{n+1}y^{-1}(-(yx)^{n+1}y^{-1}\widetilde{w}_{2q-2} - (yx)^{n+1}) - y$. We can show that $B = \widetilde{\lambda}(\widehat{r})$.
Therefore, $\widetilde{w}_{2q+1} = -4m_q b_n(y + (yx)^{n+1})y^{-1}\widetilde{w}_{2q-1} + \widetilde{\lambda}(\widehat{r})$.
Case (i) Suppose $\sum_{j=1}^{q+1} k_j \equiv 0 \pmod 4$.
Then $\sum_{j=1}^{q} k_j \equiv 3 \pmod 4$ and hence,

$$\begin{aligned} Am_q y^{-1}\widetilde{w}_{2q-1} &= -4m_q F_3(r')b_n(y + (yx)^{n+1})y^{-1}(yx)^{n+1} \\ &= -4m_q F_3(r')b_n((yx)^{n+1} - y). \end{aligned}$$

And thus, $\widetilde{\lambda}(r) = (4m_q F_3(r')b_n + F_0(\widehat{r}))(y - (yx)^{n+1})$.

Case (ii) $\sum_{j=1}^{q+1} k_j \equiv 1 \pmod 4$.
Then $\sum_{j=1}^{q} k_j \equiv 0 \pmod 4$ and, since $\widetilde{w}_{2q-1} = F_0(r')(y-(yx)^{n+1})$,

$$\begin{aligned} Am_q y^{-1}\widetilde{w}_{2q-1} &= -4m_q F_0(r')b_n(y+(yx)^{n+1})y^{-1}(y-(yx)^{n+1}) \\ &= -8m_q F_0(r')b_n y. \end{aligned}$$

Also by induction, $\widetilde{\lambda}(\widehat{r}) = F_1(\widehat{r})y$, and hence $\widetilde{\lambda}(r) = (-8m_q F_0(r')b_n + F_1(\widehat{r}))y$.

Case (iii) $\sum_{j=1}^{q+1} k_j \equiv 2 \pmod 4$.
Then $\sum_{j=1}^{q} k_j \equiv 1 \pmod 4$ and hence, $\widetilde{w}_{2q-1} = F_1(r')y$.
Therefore, $Am_q y^{-1}\widetilde{w}_{2q-1} = -4m_q F_1(r')b_n(y+(yx)^{n+1})y^{-1}y$. On the other hand, $\widetilde{\lambda}(\widehat{r}) = F_2(\widehat{r})(y+(yx)^{n+1})$, and $\widetilde{\lambda}(r) = (-4m_q F_1(r')b_n + F_2(\widehat{r}))(y+(yx)^{n+1})$.

Case (iv) $\sum_{j=1}^{q+1} k_j \equiv 3 \pmod 4$.
Then $\sum_{j=1}^{q} k_j \equiv 2 \pmod 4$ and thus, $\widetilde{w}_{2q-1} = F_2(r')(y+(yx)^{n+1})$ and

$$\begin{aligned} Am_q y^{-1}\widetilde{w}_{2q-1} &= -4m_q F_2(r')b_n(y+(yx)^{n+1})y^{-1}(y+(yx)^{n+1}) \\ &= -8m_q F_2(r')b_n(yx)^{n+1}. \end{aligned}$$

By induction, since $\widetilde{\lambda}(\widehat{r}) = F_3(\widehat{r})(yx)^{n+1}$, we have $\widetilde{\lambda}(r) = (-8m_q F_2(r')b_n + F_3(\widehat{r}))(yx)^{n+1}$. For this case, Theorem 8.3 is now proved.
From the above proof, we notice that $\widetilde{\lambda}(r)$ depends only on k_{q+1} and $\sum_{j=1}^{q+1} k_j \pmod 4$. Therefore, in the rest of our proof, it suffices to consider only the case where $(k_q, k_{q+1}) = (1,2)$ and $(1,3)$.

Case (1.2) $(k_q, k_{q+1}) = (1,2)$

We write $\widetilde{w}_{2q+1} = Am_q y^{-1}\widetilde{w}_{2q-1} + B$, where $A = -(1+y)Q_{2n}y(1+x)$, and $B = \{-(1+y)Q_{2n}y + (1+y)Q_n y - (yx)^{n+1}\}\widetilde{u}_{2q-2} - (yx)^{n+1}y^{-1}\widetilde{w}_{2q-2} - (yx)^{n+1}$. It is shown that $B = \{-(1+y)Q_{3n+1}y - (yx)^{n+1}\}\widetilde{u}_{2q-2} - (yx)^{n+1}y^{-1}\widetilde{w}_{2q-2} - (yx)^{n+1}$, which is $\widetilde{\lambda}(\widehat{r})$. Further, by (4.6)(3), we see

$$Am_q y^{-1}\widetilde{w}_{2q-1} = -8b_n m_q (yx)^{n+1}y^{-1}\widetilde{w}_{2q-1}.$$

Case (i) $\sum_{j=1}^{q+1} k_j \equiv 0 \pmod 4$.
Since $k_{q+1} = 2, \sum_{j=1}^{q} k_j \equiv 2 \pmod 4$ and hence, by induction assumption, $\widetilde{w}_{2q-1} = F_2(r')(y+(yx)^{n+1})$. Therefore,

$$\begin{aligned} \widetilde{\lambda}(r) &= -8m_q F_2(r')b_n(yx)^{n+1}y^{-1}(y+(yx)^{n+1}) + F_0(\widehat{r})(y-(yx)^{n+1}) \\ &= -8m_q F_2(r')b_n((yx)^{n+1} - y) + F_0(\widehat{r})(y-(yx)^{n+1}) \\ &= (8m_q F_2(r')b_n + F_0(\widehat{r}))(y-(yx)^{n+1}). \end{aligned}$$

Case (ii) $\sum_{j=1}^{q+1} k_j \equiv 1 \pmod 4$.
Then $\sum_{j=1}^{q} k_j \equiv 3 \pmod 4$ and $\widetilde{w}_{2q-1} = F_3(r')(yx)^{n+1}$. Therefore

$$\begin{aligned} \widetilde{\lambda}(r) &= -8m_n F_3(r')b_n(yx)^{n+1}y^{-1}(yx)^{n+1} + F_1(\widehat{r})y \\ &= (8m_n F_3(r')b_n + F_1(\widehat{r}))y. \end{aligned}$$

Case (iii) $\sum_{j=1}^{q+1} k_j \equiv 2 \pmod 4$.
Then $\sum_{j=1}^{q} k_j \equiv 0 \pmod 4$ and hence, $\widetilde{w}_{2q-1} = F_0(r')(y-(yx)^{n+1})$ and

$$\begin{aligned}\widetilde{\lambda}(r) &= -8m_q F_0(r')b_n(yx)^{n+1}y^{-1}(y-(yx)^{n+1}) + F_2(\widehat{r})(y+(yx)^{n+1})\\ &= (-8m_q F_0(r')b_n + F_2(\widehat{r}))(y+(yx)^{n+1}).\end{aligned}$$

Case (iv) $\sum_{j=1}^{q+1} k_j \equiv 3 \pmod 4$.
Then $\sum_{j=1}^{q} k_j \equiv 1 \pmod 4$ and hence, $\widetilde{w}_{2q-1} = F_1(r')y$ and

$$\begin{aligned}\widetilde{\lambda}(r) &= -8m_q F_1(r')b_n(yx)^{n+1}y^{-1}y + F_3(\widehat{r})(yx)^{n+1}\\ &= (-8m_q F_1(r')b_n + F_3(\widehat{r}))(yx)^{n+1}.\end{aligned}$$

Thus for this case, Theorem 8.3 is proved.

Case (1.3) $(k_q, k_{q+1}) = (1,3)$
Let $\widetilde{w}_{2q+1} = Am_q y^{-1}\widetilde{w}_{2q-1} + B$, where
$A = \{-(1+y)Q_{3n+1} - (yx)^{n+1}\}(x+1)$, and
$B = \{-(1+y)Q_{3n+1}y - (yx)^{n+1} - (yx)^{n+1}y^{-1} - (1+y)Q_n y + (yx)^{n+1}\}\widetilde{u}_{2q-2} - (yx)^{n+1}y^{-1}(yx)^{n+1}y^{-1}\widetilde{w}_{2q-2} = \widetilde{\lambda}(\widehat{r})$.
Further, $Am_q y^{-1}\widetilde{w}_{2q-1} = 4b_n m_q(y-(yx)^{n+1})y^{-1}\widetilde{w}_{2q-1}$. Therefore, $\widetilde{w}_{2q+1} = 4b_n m_q(y-(yx)^{n+1})y^{-1}\widetilde{w}_{2q-1} + \widetilde{\lambda}(\widehat{r})$.

Case (i) $\sum_{j=1}^{q+1} k_j \equiv 0 \pmod 4$.
Then $\sum_{j=1}^{q} k_j \equiv 1 \pmod 4$ and hence, by induction assumption, $\widetilde{w}_{2q-1} = F_1(r')y$. Since $\widetilde{\lambda}(\widehat{r}) = F_0(\widehat{r})(y-(yx)^{n+1})$, we have

$$\begin{aligned}\widetilde{\lambda}(r) &= 4m_q F_1(r')b_n(y-(yx)^{n+1})y^{-1}y + F_0(\widehat{r})(y-(yx)^{n+1})\\ &= (4m_q F_1(r')b_n + F_0(\widehat{r}))(y-(yx)^{n+1}).\end{aligned}$$

Case (ii) $\sum_{j=1}^{q+1} k_j \equiv 1 \pmod 4$.
Since $\sum_{j=1}^{q} k_j \equiv 2 \pmod 4$, we see $\widetilde{w}_{2q-1} = F_2(r')(y+(yx)^{n+1})$ and $\widetilde{\lambda}(\widehat{r}) = F_1(\widehat{r})y$, and thus

$$\begin{aligned}\widetilde{\lambda}(r) &= 4m_q F_2(r')b_n(y-(yx)^{n+1})y^{-1}(y+(yx)^{n+1}) + F_1(\widehat{r})y\\ &= (8m_q F_2(r')b_n + F_1(\widehat{r}))y.\end{aligned}$$

Case (iii) $\sum_{j=1}^{q+1} k_j \equiv 2 \pmod 4$.
Then $\sum_{j=1}^{q} k_j \equiv 3 \pmod 4$ and hence $\widetilde{w}_{2q-1} = F_3(r')(yx)^{n+1}$ and, $\widetilde{\lambda}(\widehat{r}) = F_2(\widehat{r})(y+(yx)^{n+1})$. Therefore,

$$\begin{aligned}\widetilde{\lambda}(r) &= 4m_n F_3(r')b_n(y-(yx)^{n+1})y^{-1}(yx)^{n+1} + F_2(\widehat{r})(y+(yx)^{n+1})\\ &= (4m_q F_3(r')b_n + F_2(\widehat{r}))(y+(yx)^{n+1}).\end{aligned}$$

Case (iv) $\sum_{j=1}^{q+1} k_j \equiv 3 \pmod 4$.
Then $\sum_{j=1}^{q} k_j \equiv 0 \pmod 4$ and hence $\widetilde{w}_{2q-1} = F_0(r')(y-(yx)^{n+1})$ and $\widetilde{\lambda}(\widehat{r}) =$

$F_3(\widehat{r})(yx)^{n+1}$. Thus,

$$\begin{aligned}\widetilde{\lambda}(r) &= 4m_q F_0(r')b_n(y-(yx)^{n+1})y^{-1}(y-(yx)^{n+1}) + F_3(\widehat{r})(yx)^{n+1}\\ &= (-8m_q F_0(r')b_n + F_3(\widehat{r}))(yx)^{n+1}.\end{aligned}$$

A proof of Theorem 8.3, and hence, a proof of Theorem A is now complete. □

9. Evaluation of μ.

For $r = [pk_1, 2m_1, pk_2, 2m_2, \ldots, 2m_q, pk_{q+1}]$, we proved that $\lambda_{\rho,K(r)}(-1) = \mu^2$ for some $\mu \in \mathbb{Z}[s_0]$. For convenience, we denote $\mu = \mu(r)$. In this section, we give an algorithm by which one can compute $\mu(r)$. We should note that $\mu(r) = F_j(r)$, where $j \equiv \sum_{i=1}^{q+1} k_i \pmod 4$. As we used in the previous section, let
$r' = [pk_1, 2m_1, pk_2, 2m_2, \cdots, 2m_{q-1}, pk_q]$, and
$\widehat{r} = [pk_1, 2m_1, pk_2, 2m_2, \cdots, 2m_{q-1}, p(k_q + k_{q+1})]$.
In the proof of Theorem 8.3, we have shown the following proposition.

Proposition 9.1. *The following equalities hold:*

$$\begin{aligned}&(1)\ \mu(r) = \nu\mu(r') + \mu(\widehat{r}).\\ &(2)\ \mu[0] = 0, \mu[p] = \mu[2p] = \mu[3p] = -1.\end{aligned} \tag{9.1}$$

Here $\nu = m_q b_n \sigma(k_{q+1}, M)$ and b_n is the $(1,2)$-entry of $\rho(xy)^n$, $M \equiv \sum_{j=1}^{q+1} k_j \pmod 4$, $0 \leq M \leq 3$, and $\sigma(k_{q+1}, M)$ is given by the following table:
$\sigma(1,0) = 4$, $\sigma(1,1) = -8$, $\sigma(1,2) = -4$, $\sigma(1,3) = -8$,
$\sigma(2,0) = 8$, $\sigma(2,1) = 8$, $\sigma(2,2) = -8$, $\sigma(2,3) = -8$,
$\sigma(3,0) = 4$, $\sigma(3,1) = 8$, $\sigma(3,2) = 4$, $\sigma(3,3) = -8$.

Example 9.2. Let $p = 3$ and $n = 1$, and hence $b_n = 1$.

(1) Let $r = [3, -4, 3, 2, 3]$. To evaluate $\mu(r)$, it is convenient to use the tree diagram below:

$$\begin{tikzcd}[column sep=small]
 & & & r = [3,-4,3,2,3] \arrow[dll, "\nu_1"'] \arrow[drr] & & & \\
 & r' = [3,-4,3] \arrow[dl, "\nu_2"'] \arrow[dr] & & & & \widehat{r} = [3,-4,6] \arrow[dl, "\nu_3"'] \arrow[dr] & \\
(r')' = [3] & & (\widehat{r'}) = [6] & & (\widehat{r}') = [3] & & (\widehat{\widehat{r}}) = [9]
\end{tikzcd}$$

Since, $m_1 = -2, m_2 = 1$, $k_1 = k_2 = k_3 = 1$, the weights are $\nu_1 = \sigma(1,3) = -8, \nu_2 = \sigma(1,2)(-2) = (-4)(-2) = 8$ and $\nu_3 = \sigma(2,3)(-2) = (-8)(-2) = 16$, and hence, $\mu = \nu_1\nu_2\mu[3] + \nu_1\mu[6] + \nu_3\mu[3] + \mu[9] = 55$.

(2) Let $r = [6, 2, 6, -2, 9]$. Since $m_1 = 1, m_2 = -1, k_1 = k_2 = 2, k_3 = 3$, the weights are $\nu_1 = \sigma(3,3)(-1) = (-8)(-1) = 8, \nu_2 = \sigma(2,0) = 8$ and $\nu_3 = \sigma(1,3) = -8$, and hence, $\mu = \nu_1\nu_2\mu[6] + 0 + \nu_3\mu[6] + \mu[9] = -64 + 8 - 1 = -57$.

$$
\begin{tikzcd}
 & & & {r = [6,2,6,-2,9]} \arrow[dll, "\nu_1"'] \arrow[dr] & & & \\
 & {[6,2,6]} \arrow[dl, "\nu_2"'] \arrow[dr] & & & {[6,2,15]} \arrow[r] & {[6,2,3]} \arrow[dl, "\nu_3"'] \arrow[dr] & \\
{[6]} & & {[12]} \arrow[r] & {[0]} & {[6]} & & {[9]}
\end{tikzcd}
$$

Using these recursion formulas, we can prove, for example, the following:

$$
\begin{aligned}
&(1)\ \mu[p, 2m, p] = 4mb_n - 1.\\
&(2)\ \mu[p, 2m, 2p] = 8mb_n - 1.\\
&(3)\ \mu[p, 2m, 3p] = -4mb_n.\\
&(4)\ \mu[2p, 2m, 2p] = -8mb_n.\\
&(5)\ \mu[2p, 2m, 3p] = -8mb_n - 1.\\
&(6)\ \mu[3p, 2m, 3p] = -4mb_n - 1.
\end{aligned} \tag{9.2}
$$

$$
\begin{aligned}
&(1)\ \mu[p, 2m_1, p, 2m_2, p] = -32m_1m_2b_n^2 + (8m_1 + 8m_2)b_n - 1\\
&(2)\ \mu[p, 2m_1, 2p, 2m_2, 2p] = 64m_1m_2b_n^2 - 8m_2b_n - 1.
\end{aligned} \tag{9.3}
$$

From these formulas, the following proposition is evident.

Proposition 9.3. *For any knot $K(r)$ in $H(p)$, $\mu(r) \equiv -1$ (mod 4).*

Example 9.4. (1) Let $p = 3$ and $r = [3, 4, 3, 2, 3]$. Then $\mu(r) = -41$.

(2) Let $p = 5$ and $n = 2$, then $K(19/85)$ belongs to $H(5)$.

Let s_0 be a root of $1 + 3z + z^2 = 0$. Then $b_2 = 2 + s_0$. Since $19/85 = [5, 2, 10]$, it follows from (9.2)(2) $\mu(r) = 8(2 + s_0) - 1 = 8s_0 + 15$ and $\lambda_{\rho,K(19/85)}(-1) = (8s_0 + 15)^2$.

(3) Let $p = 7$ and $n = 3$, then $K(29/217)$ belongs to $H(7)$. Let s_0 be a root of $1 + 6z + 5z^2 + z^3 = 0$. Then $b_3 = 3 + 4s_0 + s_0^2$. Since $29/217 = [7, -2, 14]$, we have from (9.2)(2) $\mu(r) = -8(3 + 4s_0 + s_0^2) - 1 = -25 - 32s_0 - 8s_0^2$ and $\lambda_{\rho,K(29/217)}(-1) = (25 + 32s_0 + 8s_0^2)^2$.

Two continued fractions $r = [pk_1, 2m_1, pk_2, 2m_2, \cdots, pk_{\ell+1}]$ and $r' = [pk'_1, 2m'_1, pk'_2, 2m'_2, \cdots, pk'_{q+1}]$ are said to be (mod 4)-*equivalent* if r is transformed into r' by a finite sequence of the following four operations and their inverses:

(1) replacement of k_i by $k_i + 4d$, $d \in \mathbb{Z}$,

(2) reduction of $[\cdots, pk_i, 0, pk_{i+1}, \cdots]$ to $[\cdots, p(k_i + k_{i+1}), \cdots]$,
(3) reduction of $[\cdots, 2m_i, 0, 2m_{i+1}, \cdots]$ to $[\cdots, 2(m_i + m_{i+1}), \cdots]$,
(4) reduction of $[\cdots, pk_r, 2m_r, 0]$ to $[\cdots, pk_r]$
(5) reduction of $[0, 2m_1, \ldots]$ to $[pk_2, 2m_2, \ldots]$.

For example, $[p, 2, 4p, -2, 2p]$ is equivalent to $[3p]$.

Computations show that the following conjecture is plausible.

Conjecture 9.5. *$\mu(r) = -1$ if and only if r is (mod 4)-equivalent to either $[p]$ or $[3p]$.*

10. Generalization and Silver-Williams Conjecture

Let $\rho : G(K(r)) \to SL(2, \mathbb{Z}[s_r]) \subset SL(2, \mathbb{C})$ be a canonical parabolic representation of $G(K(r))$ defined in Section 2, where $r = \beta/\alpha < 1$. The representation polynomial $a(z)$ of ρ has the following properties. (See [15].)

$$\begin{aligned} &(1)\ a(z) \text{ is a monic integer polynomial of degree } (\alpha-1)/2. \\ &(2)\ \text{All the roots of } a(z) = 0 \text{ are distinct and simple.} \end{aligned} \tag{10.1}$$

Let $\widetilde{\Delta}_{\rho,K(r)}(t)$ be the twisted Alexander polynomial of $K(r)$ associated to ρ. Then $\widetilde{\Delta}_{\rho,K(r)}(t)$ is a polynomial over $\mathbb{Z}[s_r]$. In order to emphasize this fact, sometimes we denote it by $\widetilde{\Delta}_{\rho,K(r)}(t|s_r)$. Let $\theta(z)$ be the minimal polynomial of s_r and $\deg\theta(z) = d$. Let $\gamma_1, \gamma_2, \cdots, \gamma_d$ be all the roots of $\theta(z) = 0$. Recently, D.Silver and S.Williams consider the integer polynomial $D_{\rho(\theta),K(r)}(t)$ defined as

$$D_{\rho(\theta),K(r)}(t) = \prod_{j=1}^{d} \widetilde{\Delta}_{\rho,K(r)}(t|\gamma_j). \tag{10.2}$$

They call it *the total $\rho(\theta)$-twisted Alexander polynomial* of K and they propose the following conjecture.

Conjecture 10.1. *[16] For any 2-bridge knot $K(r)$ and a canonical parabolic representation ρ,*
(1) $|D_{\rho(\theta),K(r)}(1)| = 2^d$ and
(2) $|D_{\rho(\theta),K(r)}(-1)| = 2^d N^2$, where $d = \deg\theta$ and N is a non-zero integer.

As they point out, $D_{\rho(\theta),K(r)}(t)$ can be evaluated as follows.
Let C be the companion matrix of the polynomial $\theta(z)$ and consider the homomorphism $\Psi : \mathbb{Z}G(K(r)) \to M_{2d,2d}(\mathbb{Z}[t^{\pm 1}])$, defined by $\Psi : x \mapsto \begin{bmatrix} E & E \\ 0 & E \end{bmatrix}$, $y \mapsto \begin{bmatrix} E & 0 \\ C & E \end{bmatrix}$, where E is the identity matrix of degree d.
It is known that

$$D_{\rho(\theta),K(r)}(t) = \det[\widetilde{\Delta}_{\rho,K(r)}(t|C)], \tag{10.3}$$

where $\widetilde{\Delta}_{\rho,K(r)}(t|C)$ is a matrix of degree $2d$ obtained from $\widetilde{\Delta}_{\rho,K(r)}(t|s_r)$ by substituting C for s_r. Computations below show that the conjecture holds for $r = 3/5, 3/7$ and $5/9$. See Example 2.3.

For $r = 3/5$, $D_{\rho(\theta),K(r)}(t) = (1-4t+t^2)^2$ and hence $D_{\rho(\theta),K(r)}(1) = 2^2$ and $D_{\rho(\theta),K(r)}(-1) = 2^2 3^2$. Note that $\theta(z) = a(z) = 1 - z + z^2$.

For $r = 3/7, \widetilde{\Delta}_{\rho,K(r)}(t) = -(4+s_r^2) + 4t - (4+s_r^2)t^2$, and hence, we have $D_{\rho(\theta),K(r)}(t) = \det[\widetilde{\Delta}_{\rho,K(r)}(t|C)] = 25 - 104t + 219t^2 - 272t^3 + 219t^4 - 104t^5 + 25t^6$, and $D_{\rho(\theta),K(r)}(1) = 2^3$ and $D_{\rho(\theta),K(r)}(-1) = 2^3 11^2$. Note $\theta(z) = a(z) = 1 + 2z + z^2 + z^3$.

For $r = 5/9, D_{\rho(\theta),K(r)}(t) = 41 - 376t + 1428t^2 - 2984t^3 + 3798t^4 - 2984t^5 + 1428t^6 - 376t^7 + 41t^8$, and hence, $D_{\rho(\theta),K(r)}(1) = 2^4$ and $D_{\rho(\theta),K(r)}(-1) = 2^4 29^2$. Note $\deg\theta = 4$.

In this section, as a simple application of our main theorem, we prove Conjecture 10.1 for a torus knot $K(1/p)$ and a knot $K(r)$ in $H(p)$.

Let $\tau : G(K(1/p)) \to SL(2,\mathbb{Z}[s_0]) \subset SL(2,\mathbb{C})$ be the canonical parabolic presentation, and $a_n(z)$ the representation polynomial of τ. The properties of $a_n(z)$ are well-studied in [15] and [18], some of which are listed below.

Proposition 10.2. *Let $p = 2n+1$. (1) $a_n(z) = \prod \chi_s(z)$, where the product runs over all odd integers s dividing $p, 3 \le s \le p$ and $\chi_s(z)$ is an irreducible, monic integer polynomial. The degree of $\chi_s(z)$ is given by $\phi(s)/2$, where $\phi(s)$ is Euler function, i.e. the number of integers $m, 1 \le m \le s$, that are relatively prime to s. In particular, if p is prime, then $\chi_p(z) = a_n(z)$.*
(2) $a_n(z) = \sum_{k=0}^{n} \binom{n+k}{2k} z^k$. (3) All the roots of $a_n(z) = 0$ are distinct and simple, and they are $-4sin^2\frac{(2k-1)\pi}{2(2k+1)}, 1 \le k \le n$, and hence all the roots are real and are in the interval $(-4, 0)$.

Example 10.3. Here are some examples of $\chi_s(z)$.
(1) $\chi_3(z) = a_1(z) = 1 + z$
(2) $\chi_5(z) = a_2(z) = 1 + 3z + z^2$
(3) $\chi_7(z) = a_3(z) = 1 + 6z + 5z^2 + z^3$
(4) $\chi_9(z) = 1 + 9z + 6z^2 + z^3$
(5) $\chi_{15}(z) = 1 + 24z + 26z^2 + 9z^3 + z^4$
(6) $\chi_{21}(z) = 1 + 48z + 148z^2 + 146z^3 + 64z^4 + 13z^5 + z^6$

Now let s_0 be a zero of $\chi_q(z), q|p, q \ge 3$. Let $r_1, r_2, \dots, r_d, d = \deg\chi_q(z) = \phi(q)/2$, be the roots of $\chi_q(z) = 0$. Then, by Proposition 2.4, the total $\tau(\chi_q)$-twisted Alexander polynomial $D_{\tau(\chi_q),K(r)}(t)$ is given by

$$D_{\tau(\chi_q),K(1/p)}(t)$$
$$= \prod_{j=1}^{d} [b_1(r_j) + b_2(r_j)t^2 + \cdots + b_n(r_j)t^{2n-2} + b_n(r_j)t^{2n} + \cdots + b_1(r_j)t^{4n-2}],$$

and hence, by (4.3)(2), we have,

$$D_{\tau(\chi_q),K(1/p)}(\pm 1)$$
$$= \prod_{j=1}^{d} [b_1(r_j) + b_2(r_j) + \cdots + b_n(r_j) + b_n(r_j) + \cdots + b_1(r_j)] = \prod_{j=1}^{d} (-2r_j^{-1}).$$

Since deg$\chi_q(z) = d$, we have $r_1 r_2 \cdots r_d = (-1)^d$ and hence $D_{\tau(\chi_q),K(1/p)}(\pm 1) = 2^d$. This proves Conjecture 10.1 for $K(1/p)$.

Similar arguments work for $K(r)$ in $H(p)$.

Let $\rho = \tau\varphi$ be the canonical parabolic presentation of $G(K(r))$, $\rho : G(K(r)) \to G(K(1/p)) \to SL(2, \mathbb{Z}[s_0])$.
As before, we assume that s_0 is a zero of $\chi_q(z), q|p$, and $r_j, 1 \leq j \leq d$, are roots of $\chi_q(z) = 0$. Then $\widetilde{\Delta}_{\rho,K(r)}(t|s_0) = \lambda_{\rho,K(r)}(t|s_0)\widetilde{\Delta}_{\tau,K(1/p)}(t|s_0)$, and $D_{\rho(\chi_q),K(r)}(t) = \{\prod_{j=1}^d \lambda_{\rho,K(r)}(t|r_j)\} D_{\tau(\chi_q),K(1/p)}(t)$.
Now by Theorem A, Propositions 2.4 and 4.3(III)(3), we have $D_{\rho(\chi_q),K(r)}(1) = D_{\tau(\chi_q),K(1/p)}(1) = 2^d$.
Further, if we write $\lambda_{\rho,K(r)}(-1|r_j) = \mu_j^2$, then $D_{\rho(\chi_q),K(r)}(-1) = (\mu_1\mu_2 \cdots \mu_d)^2 2^d$.
This proves Conjecture 10.1 for the total $\rho(\chi_q)$-twisted Alexander polynomial of $K(r)$ in $H(p)$.

Proposition 10.4. *For a knot $K(1/p)$, the total $\rho(\chi_q)$-twisted Alexander polynomial $D_{\rho(\chi_q),K(r)}(t)$ can be determined by the following three formulas. Let $p = 2n+1$.*
(1) If q is a divisor of p, say $p = vq, v \geq 3$, then
$D_{\tau(\chi_q),K(1/p)}(t) = (1 - t^{2q} + t^{4q} - \cdots + t^{2(v-1)q})^{d_q} D_{\tau(\chi_q),K(1/p)}(t)$, *where $d_q =$deg$\chi_q(t)$.*
(2) $\prod D_{\tau(\chi_u),K(1/p)}(t) = (1 + t^2)(1 + t^{4n+2})^{n-1}$, *(10.4)*
where the product runs over all divisors $u (\neq 1)$ of p.
(3) If p is a prime, then $D_{\tau(\chi_p),K(1/p)}(t) = (1 + t^2)(1 + t^{4n+2})^{n-1}$.

Since Proposition 10.4(1) is an easy consequence of Proposition 3.5 and Proposition 10.4(3) follows from Proposition 10.4(2), only a proof of Proposition 10.4(2) will be given in Appendix (III).

Example 10.5. (1) Let $p = 9$ and $n = 4$. Then $a_4(z) = \chi_3(z)\chi_9(z)$. First, by Proposition 10.4(3) $D_{\tau(\chi_3),K(1/3)}(t) = 1 + t^2$, and by Proposition 10.4(1), we see $D_{\tau(\chi_3),K(1/9)}(t) = (1 + t^2)(1 - t^6 + t^{12})$. Further, by Proposition 10.4(2), $D_{\tau(\chi_3),K(1/9)}(t) D_{\tau(\chi_9),K(1/9)}(t) = (1 + t^2)(1 + t^{18})^3$, and hence, $D_{\tau(\chi_9),K(1/9)}(t) = (1 + t^2)(1 + t^{18})^3/(1 + t^2)(1 - t^6 + t^{12}) = (1 + t^{18})^2(1 + t^6)$.
(2) Let $p = 15$ and $n = 7$. Then $a_7(z) = \chi_3(z)\chi_5(z)\chi_{15}(z)$ and

$$\begin{aligned} D_{\tau(\chi_3),K(1/15)}(t) &= \lambda_{\tau(\chi_3),K(1/15)}(t) D_{\tau(\chi_3),K(1/3)}(t) \\ &= (1 + t^2)(1 - t^6 + t^{12} - t^{18} + t^{24}), \end{aligned}$$

$$\begin{aligned} D_{\tau(\chi_5),K(1/15)}(t) &= \{\lambda_{\tau(\chi_5),K(1/15)}(t)\}^2 D_{\tau(\chi_5),K(1/5)}(t) \\ &= (1 - t^{10} + t^{20})^2(1 + t^2)(1 + t^{10}). \end{aligned}$$

Since $\prod_{j=3,5,15} D_{\tau(\chi_j),K(1/15)}(t) = (1 + t^2)(1 + t^{30})^6$, we have

$$\begin{aligned} D_{\tau(\chi_{15}),K(1/15)}(t) &= (1 + t^2)(1 + t^{30})^6 / D_{\tau(\chi_3),K(1/15)}(t) D_{\tau(\chi_5),K(1/15)}(t) \\ &= (1 - t^2 + t^4)(1 + t^{10})(1 + t^{30})^3. \end{aligned}$$

Next we discuss a generalization of our main theorem.
Suppose there is an epimorphism φ from $G(K(r))$ to $G(K(r_0))$. Using a canonical parabolic representation ρ of $G(K(r_0))$ into $SL(2,\mathbb{C})$, we can define the the twisted Alexander polynomials $\widetilde{\Delta}_{\rho\varphi,K(r)}(t)$ and $\widetilde{\Delta}_{\rho,K(r_0)}(t)$ associated to $\rho\varphi$ and ρ, respectively. Since $\widetilde{\Delta}_{\rho,K(r_0)}(t)$ divides $\widetilde{\Delta}_{\rho\varphi,K(r)}(t)$, the quotient $\lambda_{\rho,K(r)}(t)$ is well-defined. The following conjecture is a generalization of our main theorem.

Conjecture 10.6. *(1)* $\lambda_{\rho,K(r)}(1) = 1$*, and*
(2) $\lambda_{\rho,K(r)}(-1) = \mu^2$ *for some* $\mu \in \mathbb{Z}[s_{r_0}]$*.*

In fact, there is an epimorphism $\varphi : G(K(63/115)) \to G(K(3/5))$ and we have the twisted Alexander polynomial of $K(63/115)$
$\widetilde{\Delta}_{\rho\varphi,K(63/115)}(t) = \lambda_{\rho,K(63/115)}(t)\widetilde{\Delta}_{\rho,K(3/5)}(t)$, where $\lambda_{\rho,K(63/115)}(t) = (3-w) - (16-8w)t + (33-34w)t^2 - (40-76w)t^3 + (41-98w)t^4 - (40-76w)t^5 + (33-34w)t^6 - (16-8w)t^7 + (3-w)t^8$, and w is a primitive third root of 1. We see then $\lambda_{\rho,K(63/115)}(1) = 1$ and $\lambda_{\rho,K(63/115)}(-1) = 225 - 336w = (17-8w)^2$.

Finally, we give a few remarks on the representation polynomials. Let $f(z)$ and $g(z)$, respectively, be the representation polynomials of $\rho\varphi$ and ρ. Then $g(z)$ divides $f(z)$. However, the converse seems quite likely to hold, and therefore, we propose the following conjecture.

Conjecture 10.7. *Let* $f_1(z)$ *and* $f_2(z)$*, respectively, be the representation polynomials of the canonical parabolic representations* $\rho_1\colon G(K(r_1)) \to SL(2,\mathbb{C})$ *and* $\rho_2\colon G(K(r_2)) \to SL(2,\mathbb{C})$*. If* $f_2(z)$ *divides* $f_1(z)$*, then there exists an epimorphism from* $G(K(r_1))$ *to* $G(K(r_2))$*.*

It is proven [11],[1] that Conjecture 10.7 holds if $r_2 = 1/p$ or equivalently, if $\chi_p(z)$ divides $f_1(z)$, then there exists an epimorphism from $G(K(r_1))$ to $G(K(1/p))$.

Remark 10.8. Very recently we learned [17] that D. Silver and S. Williams proved Conjecture 10.1 (1) for any 2-bridge knot $K(r)$.

11. Appendix

In Appendix, we discuss four topics.

(I) Outline of the proof of Proposition 2.4.

Consider a Wirtinger presentation of $G(K(1/p))$ given by (2.2):

$G(K(1/p)) = \langle x, y | R_0 = WxW^{-1}y^{-1}\rangle$, where $p = 2n+1$ and $W = (xy)^n$. Then

$\frac{\partial R_0}{\partial x} = (1-y)Q_{n-1} + (xy)^n$ and hence,

$$\begin{aligned} D &= (\frac{\partial R_0}{\partial x})^{\Phi} \\ &= \begin{bmatrix} 1-t & 0 \\ -s_0 t & 1-t \end{bmatrix} \begin{bmatrix} \sum_{k=0}^{n-1} a_k t^{2k} & \sum_{k=0}^{n-1} b_k t^{2k} \\ \sum_{k=0}^{n-1} c_k t^{2k} & \sum_{k=0}^{n-1} d_k t^{2k} \end{bmatrix} + \begin{bmatrix} 0 & b_n t^{2n} \\ c_n t^{2n} & d_n t^{2n} \end{bmatrix} \\ &= \begin{bmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{bmatrix}, \text{ where} \end{aligned}$$

$$\begin{aligned} h_{11} &= (1-t)\sum_{k=0}^{n-1} a_k t^{2k}, \\ h_{12} &= (1-t)\sum_{k=0}^{n-1} b_k t^{2k} + b_n t^{2n}, \\ h_{21} &= -s_0 t \sum_{k=0}^{n-1} a_k t^{2k} + (1-t)\sum_{k=0}^{n-1} c_k t^{2k} + c_n t^{2n}, \text{ and} \\ h_{22} &= -s_0 t \sum_{k=0}^{n-1} b_k t^{2k} + (1-t)\sum_{k=0}^{n-1} d_k t^{2k} + d_n t^{2n}. \end{aligned}$$

Since for $j \geq 1, a_0 + a_1 + \cdots + a_{j-1} = b_j$ and $s_0 b_j = c_j$, we see that

$$\sum_{k=0}^{n-1} a_k t^{2k} = \sum_{k=1}^{n} b_k t^{2k-2}(1-t^2) + b_n t^{2n}, \text{ and hence}$$

$$h_{2,1} = (1-t)\{-\sum_{k=1}^{n} c_k t^{2k-1}(1+t) + \sum_{k=0}^{n-1} c_k t^{2k} + c_n t^{2n}\} = -(1-t)\sum_{k=1}^{n} c_k t^{2k-1}.$$

Thus the first column is divisible by $1-t$ and hence,

$$\det D/(1-t) = \det \begin{bmatrix} h'_{1,1} & h_{1.2} \\ h'_{2,1} & h_{2,2} \end{bmatrix}, \text{ where}$$

$$h'_{1,1} = \sum_{k=0}^{n-1} a_k t^{2k} \text{ and } h'_{2,1} = -\sum_{k=1}^{n} c_k t^{2k-1}.$$

Now subtract the first column multiplied through t from the second column so that we have

$$\begin{aligned} h_{1,2} - t h'_{1,1} &= b_n t^{2n} + (1-t)\sum_{k=1}^{n-1} b_k t^{2k} - \sum_{k=1}^{n} b_k t^{2k-1}(1-t^2) - b_n t^{2n+1} \\ &= -(1-t)\sum_{k=1}^{n} b_k t^{2k-1}. \end{aligned}$$

Similarly, noting $c_k + d_k = a_k$ for $k \geq 0$, we obtain

$$\begin{aligned} h_{2,2} - th'_{2,1} &= -t\sum_{k=0}^{n-1} c_k t^{2k} + (1-t)\sum_{k=0}^{n-1} d_k t^{2k} + d_n t^{2k} + t\sum_{k=1}^{n} c_k t^{2k-1} \\ &= (1-t)\sum_{k=0}^{n-1} a_k t^{2k}. \end{aligned}$$

Therefore

$$\det D/(1-t)^2 = \det\begin{bmatrix} \sum_{k=0}^{n-1} a_k t^{2k} & -\sum_{k=1}^{n} b_k t^{2k-1} \\ -\sum_{k=1}^{n} c_k t^{2k-1} & \sum_{k=0}^{n-1} a_k t^{2k} \end{bmatrix}, \text{ and hence}$$

$$\widetilde{\Delta}_{\rho,K(1/p)}(t) = (\sum_{k=0}^{n-1} a_k t^{2k})^2 - (\sum_{k=1}^{n} b_k t^{2k-1})(\sum_{k=1}^{n} c_k t^{2k-1}).$$

Proposition 2.4, then, follows from (A.1) below:

$$\text{For } k = 1, 2, \ldots, n,\ b_k = \sum_{i+j=k-1} a_i a_j - \sum_{i+j=k} b_i c_j. \tag{A.1}$$

(II) Proof of Proposition 3.5.

We prove

$$\lambda_{\rho,K(1/pq)}(t) = \Delta_{K(1/q)}(t^{2p}). \tag{A.2}$$

Let $p = 2n+1$ and $q = 2m+1$. Let $G(K(1/pq)) = \langle x, y | R_{pq}\rangle$ and $G(K(1/p)) = \langle x, y | R_0\rangle$ be Wirtinger presentations, where $R_{pq} = (xy)^{2mn+m+n} x (xy)^{-(2mn+m+n)} y^{-1}$ and $R_0 = (xy)^n x (xy)^{-n} y^{-1}$. We must express R_{pq} as a product of conjugates of R_0. In fact, we prove:

Lemma A.1. $R_{pq} = R_0^{\tau_m}$, *where*

$$\tau_m = \sum_{k=0}^{m} (xy)^{kp} - \sum_{k=0}^{m-1} (xy)^{kp} (xy)^n x. \tag{A.3}$$

Proof. We prove (A.3) by induction on m. If $m = 1$, then $R_{pq} = (xy)^{3n+1} x (xy)^{-(3n+1)} y^{-1} = R_0^{\tau_1}$, where $\tau_1 = (xy)^{2n+1} - (xy)^n x + 1$. Thus (A.3) holds. Now inductively, consider τ_{m+1}. Applying the previous argument repeatedly, we obtain

$$\begin{aligned} R_{p(2m+3)} &= (xy)^{mp+3n+1} x (xy)^{-(mp+3n+1)} y^{-1} \\ &= (xy)^{mp+p}\{(xy)^n x (y^{-1}x^{-1})^n y^{-1}\} x^{-1} (y^{-1}x^{-1})^{mp+2n} y^{-1} \\ &= R_0^{(xy)^{(m+1)p}} (xy)^{(m+1)p} x^{-1} (y^{-1}x^{-1})^{mp+2n} y^{-1} \\ &= R_0^{(xy)^{(m+1)p}} (xy)^{mp+n} x \{y (xy)^n x^{-1} (y^{-1}x^{-1})^n\} (y^{-1}x^{-1})^{mp+n} y^{-1} \\ &= R_0^{(xy)^{(m+1)p}} R_0^{-(xy)^{mp+n}x} (xy)^{mp+n} x (y^{-1}x^{-1})^{mp+n} y^{-1} \\ &= R_0^{(xy)^{(m+1)p} - (xy)^{mp+n}x} R_0^{\tau_m}, \text{ and hence} \end{aligned}$$

$\tau_{m+1} = (xy)^{(m+1)p} - (xy)^{mp}(xy)^n x + \tau_m$. This proves (A.3). □

Now to evaluate $\lambda_{\rho,K(1/pq)}(t)$, we compute $\Phi(\tau_m)$ that is given as follows. Since $\Phi((xy)^{kp}) = \begin{bmatrix} (-1)^k & 0 \\ 0 & (-1)^k \end{bmatrix} t^{2kp}$ and $\Phi[(xy)^n x] = \begin{bmatrix} 0 & b_n \\ c_n & 0 \end{bmatrix} t^p$, we have

$$\begin{aligned}
\Phi(\tau_m) &= \sum_{k=0}^{m} \begin{bmatrix} (-1)^k & 0 \\ 0 & (-1)^k \end{bmatrix} t^{2kp} - \sum_{k=0}^{m-1} \begin{bmatrix} 0 & (-1)^k b_n \\ (-1)^k c_n & 0 \end{bmatrix} t^{(2k+1)p} \\
&= \begin{bmatrix} \displaystyle\sum_{k=0}^{m} (-1)^k t^{2kp} & \displaystyle -\sum_{k=0}^{m-1} (-1)^k b_n t^{(2k+1)p} \\ \displaystyle -\sum_{k=0}^{m-1} (-1)^k c_n t^{(2k+1)p} & \displaystyle\sum_{k=0}^{m} (-1)^k t^{2kp} \end{bmatrix}.
\end{aligned}$$

Since $b_n c_n = -1$, we see

$$\begin{aligned}
\det[\Phi(\tau_m)] &= \left\{ \sum_{k=0}^{m} (-1)^k t^{2kp}]^2 + \sum_{k=0}^{m-1} (-1)^k t^{(2k+1)p} \right\}^2 \\
&= \sum_{k=0}^{2m} (-1)^k t^{2kp} \\
&= \Delta_{K(1/q)}(t^{2p}).
\end{aligned}$$

This proves (A.2).

(III) Sketch of the proof of Proposition 10.4(2).

Denote $x_j = t^{2j} + t^{4n-2j-2}$, $0 \le j \le n-1$. Then $\widetilde{\Delta}_{\rho,K(1/p)}(t)$ can be written as $\widetilde{\Delta}_{\rho,K(1/p)}(x_0, \cdots, x_{n-1}) = b_1 x_0 + b_2 x_1 + b_3 x_2 + \cdots + b_n x_{n-1}$.
We use the following easy formula proved in [18].

$$\text{For } k \ge 1, b_k = \sum_{j=0}^{k-1} \binom{k+j}{2j+1} s_0^j, \text{ where } s_0 \text{ is a root of } a_n(z). \tag{A.3}$$

Let $C = [c(i,j)]_{1 \le i,j \le n}$ be the companion matrix of $a_n(z)$.
Only non-zero entries of C are:

(1) For $i = 1, 2, \cdots, n, c(i,n) = -\binom{n+i-1}{2(i-1)}$
(2) For $1 \le i \le n-1, c(i+1, i) = 1$. (A.4)

Let $C^k = [c_k(i,j)]_{1 \le i,j \le n}$. Then a straightforward calculation verifies the following lemma.

Lemma A.2. *Let $1 \le k \le n-1$.*
(1) For $1 \le i \le n-k, c_k(k+i, i) = 1$.
(2) For $1 \le i \le n, 1 \le k \le n-1, 1 \le j \le k-1, c_k(i, n-k+j) = c_{k-1}(i, n-k+j+1) = c_{k-2}(i, n-k+j+2) = \cdots = c_{j+1}(i, n-1) = c_j(i,n)$.
(3) For $k \ge 2, c_k(1,n) = c_1(1,n)c_{k-1}(n,n)$, and for $i \ge 2$ and $k \ge 2$, $c_k(i,n) = c_{k-1}(i-1,n) + c_1(i,n)c_{k-1}(n,n)$.
(4) Other values of $c_k(i,j)$ are 0.

Let $B_k = \displaystyle\sum_{j=0}^{k-1} \binom{k+j}{2j+1} C^j$ and $D = \displaystyle\sum_{j=1}^{n} B_j x_{j-1} = [d(i,j)]_{1\leq i,j\leq n}$. Since $a_n(z)$ is separable, det$D$ is the LHS of (10.4). We determine $d(i,j)$. Since the following three lemmas are easily proven, we omit the details.

Lemma A.3. *(1) For $1 \leq i < j \leq n$,*

$$d(i,j) = \sum_{m=1}^{j-1} \sum_{k=0}^{j-2-m+1} \binom{2n-2j+2m+1+k}{2n-2j+2m+1} c_m(i,n) x_{n-j+m+k},$$

(2) For $1 \leq i \leq n$,

$$d(i,i) = \sum_{k=1}^{n} k x_{k-1} + \sum_{m=1}^{i-1} \sum_{k=0}^{i-m-1} \binom{2n-2i+2m+1+k}{2n-2i+2m+1} c_m(i,n) x_{n-i+m+k},$$

(3) For $1 \leq j < i \leq n$,

$$d(i,j) = \sum_{m=1}^{j-1} \sum_{k=0}^{i-m-1} \binom{2n-2j+2m+1+k}{2n-2j+2m+1} c_m(i,n) x_{n-j+m+k} + \sum_{k=0}^{n-i+j-1} \binom{2i-2j+1+k}{2i-2j+1} x_{i-j+k}.$$

(4) For $1 \leq j \leq n-1$,

$$d(n,j) = \sum_{m=1}^{j-1} \sum_{k=0}^{j-m-1} \binom{2n-2j+2m+1+k}{2n-2j+2m+1} c_m(i,n) x_{n-j+m+k} + \sum_{k=0}^{j-1} \binom{2n-2j+1+k}{2n-2j+1} x_{n-j+k}.$$

In particular, $d(n,1) = x_{n-1}$.

There are some relations among entries of D.

Lemma A.4. *For $2 \leq i,j \leq n$*
(1) $d(1,j) = c_1(1,n) d(n,j-1)$,
(2) $d(i,j) = d(i-1,j-1) + c_1(i,n) d(n,j-1)$.

Note that for $k \neq n-1, x_k t^{-2}(1-t^2)^2 = x_{k-1} - 2x_k + x_{k+1}$ and $x_{n-1} t^{-2}(1-t^2)^2 = x_{n-2} - x_{n-1}$.

Using this lemma, we can prove:

Lemma A.5. *$d(i,i)t^{-2}(1-t^2)^2 = d(i,i+1) + t^{-2}(1+t^{4n+2})$, and if $j \neq i$, then $d(i,j)t^{-2}(1-t^2)^2 = d(i,j+1)$.*

Now consider $D = [d(i,j)]_{1\leq i,j\leq n}$. First subtract the $(n-1)^{\text{st}}$ column multiplied through $t^{-2}(1-t^2)^2$ from the n^{th} column. Then by Lemma A.4, all the entries of the resulting n^{th} column are 0 except the $(n-1,n)$ entry that is $-t^{-2}(1+t^{4n+2})$.

Successive applications of the same operation applied on the $(j-1)^{\mathrm{st}}$ column and the j^{th} column transform D into a new matrix $\widehat{D} = \left[\widehat{d}(i,j)\right]_{1\le i,j\le n}$, where the off diagonal entries $\widehat{d}(i,i+1)$ are $t^{-2}(1+t^{4n+2})$ and $\widehat{d}(i,n) = d(i,n), 1 \le i \le n$ and all the rest is 0. Thus

$$\begin{aligned}\det D &= \det \widehat{D}\\ &= \{-t^{-2}(1+t^{4n+2})\}^{n-1}(-1)^{n-1}x_{n-1}\\ &= t^{-(2n-2)}(1+t^{4n+2})^{n-1}t^{2n-2}(1+t^2)\\ &= (1+t^2)(1+t^{4n+2})^{n-1}.\end{aligned}$$

(IV) Alternative characterization of r for $K(r)$ in $H(p)$.

Definition A.6. Let α and β be co-prime odd integers with $0<|\beta|<\alpha$, and p an odd integer.

(I) We say that $r=\beta/\alpha$ is p-*expandable* if r has a continued fraction expansion of the form:

$$r=[pk_1,2m_1,pk_2,2m_2,\ldots], \text{ where } k_i,m_i\in\mathbb{Z}\setminus\{0\}.$$

(II) We know that r has a unique continued fraction

$$r=[2a_1,2a_2,\ldots,2a_\ell,c], \text{ where } c\ne\pm1.$$

Then we inductively define r to be p-*admissible* by the following:

(a) $[c]$ is p-admissible if and only if $c\equiv p \mod 2p$,

(b) $[2a_1,c]$ is never p-admissible, and

(c) Let $r=[2a_1,2a_2,x,\ldots]$, where $x,\ldots$ denotes $2a_3,\ldots$ or c.

Then r is p-admissible if and only if one of the following is satisfied:

(i) $2a_1\equiv 0 \mod 2p$ and $[x,\ldots]$ is p-admissible.

(ii) $2a_1\equiv p+1 \mod 2p, 2a_2=2$, and $[x-(p+1),\ldots]$ is p-admissible.

(iii) $2a_1\equiv p-1 \mod 2p, 2a_2=-2$, and $[x-(p-1),\ldots]$ is p-admissible.

Example A.7. Let $r=12225937/33493827$.

Then r is both 3-expandable and 3-admissible, since

$$\begin{aligned}r &= [3,4,6,-4,9,6,18,-2,-3,4,6]\\ &= [2,-2,-2,-2,6,2,2,2,10,6,18,-2,-4,-2,-2,-2,5].\end{aligned}$$

Remark A.8. (1) Let p be an odd integer. Then both of the denominator and numerator of $r=[pk_1,2m_1,pk_2,2m_2,\ldots]$ are odd if and only if (i) the length of expansion is odd and (ii) total of k_i's is odd. (2) If both of the denominator and numerator of r is odd, then the reduction in Definition A.6. (c) preserves that property.

Lemma A.9. For continued fractions, we have the following equalities:

(1) $[\ldots,a,2,b,\ldots]=[\ldots,a-1,-2,b-1,\ldots]$

(2) $[\ldots,a,\underbrace{2,2,\ldots,2}_{k},b,\ldots]=[\ldots,a-1,-(k+1),b-1,\ldots]$

Theorem A.10. *Let α and β be co-prime odd integers with $0 < |\beta| < \alpha$, and p an odd integer. Then $r = \beta/\alpha$ is p-admissible if and only if r is p-expandable.*

Proof. (Proof of '$\Rightarrow$') Suppose $r = [2a_1, 2a_2, \ldots, 2a_\ell, c]$ is p-admissible. We prove that r is p-expandable by induction on the length of expansion. First, if $[c]$ is p-admissible, then $c = 2pn+p$ for some $n \in \mathbb{Z}$ and hence r is p-expandable. Next, if $r = [2a_1, c]$, r is not p-admissible and there is nothing to prove. Let $r = [2a_1, 2a_2, x, \ldots]$, where x denotes $2a_3$ or c.

Case 1, $2a_1 = 2pn$ for some n: Here, $[x, \ldots]$ is p-admissible, and by induction hypothesis, $[x, \ldots]$ is p-expandable. So, $r = [2pn, 2a_2, x, \ldots]$ is also p-expandable.

Case 2, $(2a_1, 2a_2) = (2pn + (p+1), 2)$ for some n: Here, $[x - (p+1), \ldots]$ is p-admissible, and hence by induction hypothesis, $[x - (p+1), \ldots]$ is p-expandable. Then $[p(2n+1), -2, x - (p+1) + p, \ldots]$ is also p-expandable. Since

$$\begin{aligned} &[p(2n+1), -2, x - (p+1) + p, \ldots] \\ &= [p(2n+1) + 1, 2, x - (p+1) + p + 1, \ldots] \\ &= r, \end{aligned}$$

we see that r is p-expandable.

Case 3, $(2a_1, 2a_2) = (2pn + (p-1), -2)$: This case is similar to Case 2.

(Proof of '$\Leftarrow$') Suppose that the length of expansion is 1, i.e., $r = [pk_1]$. Since α and β are odd, both p and k_1 are odd. Therefore, writing $k_1 = 2q+1$, we see that $pk_1 = p(2q+1) \equiv p \mod 2p$, and hence r is p-admissible. The length of expansion is never equal to 2, since if so, $r = \frac{1}{pk - 1/2m} = \frac{2m}{2pkm - 1}$ and hence β would be even. Let $r = [pk_1, 2m_1, pk_2, \ldots]$.

Case 1, k_1 is even: Here, we can write $pk_1 = 2pq$, and hence it suffices to show that $[pk_2, \ldots]$ is p-admissible, which is true since, by Remark A.8, we can use the induction hypothesis.

Case 2.1, k_1 is odd and $m_1 > 0$: Write $k_1 = 2q+1$, then we have

$$\begin{aligned} r &= [2qp + p, 2m_1, pk_2, \ldots] \\ &= [2qp + p - 1, \underbrace{-2, \ldots, -2}_{2m_1 - 1},\ pk_2 - 1, \ldots] \\ &= [\{2qp + p - 1, -2\}, \underbrace{\{-2, -2\}, \ldots, \{-2, -2\}}_{m_1 - 1},\ pk_2 - 1, \ldots]. \end{aligned}$$

(Braces are inserted just for the sake of pairing.) Then we further see that r is p-admissible if and only if so is

$$[\{-2 - (p-1), -2\}, \underbrace{\{-2, -2\}, \ldots, \{-2, -2\}}_{m_1 - 2}, pk_2 - 1, \ldots].$$

Since $-2 - (p-1) \equiv p - 1 \mod 2p$, we see that r is p-admissible if and only if so is $[\{-2 - (p-1), -2\}, \underbrace{\{-2, -2\}, \ldots, \{-2, -2\}}_{m_1 - 3}, pk_2 - 1, \ldots]$. Repeatedly r is p-admissible if and only if so is $[-(p-1) + pk_2 - 1, \ldots]$, which is p-expandable since

$-(p-1)+pk_2-1=p(k_2+1)$. Now, k_1+k_2 and k_2+1 have the same parity and hence, by Remark A.8, we can use the induction hypothesis to see that r is p-admissible.

Case 2.2, k_1 is odd and $m_1<0$: This case is similar to Case 2.1.

This completes the proof of Theorem A.10. □

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